Vertical operators on the Bergman space on the upper half-plane

This text is a rough draft.

Objectives. Establish criteria of "vertical" operators (in other words, operators that are invariant under horizontal shifts) on the Bergman space on the upper half-plane.

Requirements. Bergman space of the analytic square integrable functions on the upper half-plane, Berezin transform, shift operator, Fourier transform, Laplace transform.

Bergman kernel on the upper half-plane (short review)

Fact 1 (evaluation functional is bounded)**.** Given a point $w \in \Pi$, denote by eval_w the evaluation functional at w on the Bergman space $\mathcal{A}^2(\Pi)$:

$$\operatorname{eval}_w(f) \coloneqq f(w).$$

It is known that this functional is bounded.

Definition 1 (Bergman kernel on the upper half-plane)**.** Given a point $w \in \Pi$, denote by $K_{\Pi,w}$ the function corresponding to the evaluation functional at the point w (this function is called sometimes the *Bergman kernel* of the space $\mathcal{A}^2(\Pi)$):

$$\forall f \in \mathcal{A}^2(\Pi) \qquad \operatorname{eval}_w(f) = \langle K_{\Pi,w}, f\rangle.$$

Fact 2 (explicit formula for the Bergman kernel on the upper half-plane)**.** Let $w \in \Pi$. It is known that

$$K_{\Pi,w}(\xi) = -\frac{1}{\pi(\overline{w} - \xi)^2}.$$

Exercise 1 (norm of the Bergman kernel on the upper half-plane)**.** Let $w \in \Pi$. Calculate the norm $\|K_{\Pi,w}\|_2$ of the function $K_{\Pi,w}$.

Berezin transform of bounded linear operators (short review)

Definition 2 (Berezin transform of a bounded linear operator)**.** Let $S\colon \mathcal{A}^2(\Pi) \to \mathcal{A}^2(\Pi)$ be a bounded linear operator. The *Berezin transform* of S is defined by

$$\mathcal{B}(S)(w) := \frac{\langle S K_{\Pi,w}, K_{\Pi,w}\rangle}{\langle K_{\Pi,w}, K_{\Pi,w}\rangle}.$$

Exercise 2. Prove that the Berezin transform of bounded linear operators is a linear map:

$$\mathcal{B}(S_1 + S_2)(w) =$$

$$\mathcal{B}(\lambda S)(w) =$$

Exercise 3. Let $S\colon \mathcal{A}^2(\Pi) \to \mathcal{A}^2(\Pi)$ be a bounded linear operator and S^* be its adjoint operator. Prove that $\mathcal{B}(S^*) = \overline{\mathcal{B}(S)}$.

$$\mathcal{B}(S^*)(w) =$$

Exercise 4. Find a proof of the fact that the Berezin transform of bounded linear operators is an injective map: if $\mathcal{B}(S) = 0$, then $S = 0$.

Multiplication operator on the real line

Definition 3. Let $g \in L^\infty(\mathbb{R})$. Define $M_g \in \mathcal{L}(L^2(\mathbb{R}))$ by

$$(M_g f)(x) \coloneqq g(x) f(x).$$

Exercise 5 (product of the multiplication operators)**.** Let $g_1, g_2 \in L^\infty(\mathbb{R})$. Calculate the product of the operators M_{g_1} and M_{g_2}.

Exercise 6 (multiplication operators commute)**.** Let $g_1, g_2 \in L^\infty(\mathbb{R})$. Prove that

$$M_{g_1} M_{g_2} = M_{g_2} M_{g_1}.$$

Definition 4. Given a number $h \in \mathbb{R}$ denote by Θ_h the following function $\mathbb{R} \to \mathbb{C}$:

$$\Theta_h(x) \coloneqq \mathrm{e}^{\mathrm{i}\, hx}.$$

Fact 3 (criterion of multiplication operator on the real line)**.** Let $S \in \mathcal{L}(L^2(\mathbb{R}))$. Then the following conditions are equivalent:

(a) S is invariant under multiplication by Θ_h for all $h \in \mathbb{R}$:

$$\forall h \in \mathbb{R} \qquad S M_{\Theta_h} = M_{\Theta_h} S.$$

(b) S is the multplication operator by a bounded function:

$$\exists m \in L^\infty(\mathbb{R}) \qquad S = M_m.$$

Exercise 7. Prove that (b) implies (a).

Exercise 8. Find in the literature a proof that (a) implies (b). Write here an exact reference to the proof. Study the proof.

Multiplication operator on the positive half-line

Definition 5. Let $\sigma \in L^\infty(\mathbb{R}_+)$. Define $M_\sigma \in \mathcal{L}(L^2(\mathbb{R}_+))$ by

$$(M_\sigma f)(x) \coloneqq \sigma(x) f(x).$$

Exercise 9 (product of the multiplication operators)**.** Let $\sigma_1, \sigma_2 \in L^\infty(\mathbb{R}_+)$. Calculate the product of the operators M_{σ_1} and M_{σ_2}.

Definition 6. Given a number $h \in \mathbb{R}$ denote by Θ_h^+ the restriction of Θ_h onto the positive half-line:

$$\Theta_h^+(x) \coloneqq \mathrm{e}^{\mathrm{i}\,hx} \qquad (x \in \mathbb{R}_+).$$

Exercise 10 (criterion of multiplication operator on the positive half-line)**.** Let $S \in \mathcal{L}(L^2(\mathbb{R}_+))$. Prove that the following conditions are equivalent:

(a) S is invariant under multiplication by Θ_h^+ for all $h \in \mathbb{R}$:

$$\forall h \in \mathbb{R} \qquad S M_{\Theta_h^+} = M_{\Theta_h^+} S.$$

(b) S is the multiplication operator by a bounded function:

$$\exists \sigma \in L^\infty(\mathbb{R}_+) \qquad S = M_\sigma.$$

Vertical operators on the positive half-plane

Definition 7 (horizontal shift operator)**.** Let $h \in \mathbb{R}$. Define $H_h\colon A^2(\Pi) \to A^2(\Pi)$ by

$$(H_h f)(w) \coloneqq f(w-h).$$

Exercise 11. Let $h \in \mathbb{R}$ and $w \in \Pi$. Calculate $H_h K_{\Pi,w}$.

$$(H_h K_{\Pi,w})(z) =$$

The following fact is taken from papers of N. Vasilevski.

Fact 4. Define $R\colon \mathcal{A}^2(\Pi) \to L^2(\mathbb{R}_+)$ by

$$R\colon \mathcal{A}^2(\Pi) \to L^2(\mathbb{R}_+), \qquad (R\phi)(x) = \sqrt{x}\frac{1}{\sqrt{\pi}} \int\limits_{\Pi} \phi(w)\, \mathrm{e}^{-\mathrm{i}\,\overline{w}x}\, d\mu(w),$$

This operator R is an isometrical isomorphism, and its inverse $R^*\colon L^2(\mathbb{R}_+) \to \mathcal{A}^2(\Pi)$ is given by:

$$(R^* f)(z) = \frac{1}{\sqrt{\pi}} \int\limits_{\mathbb{R}_+} \sqrt{\xi} f(\xi)\, \mathrm{e}^{\mathrm{i}\,z\xi}\, d\xi.$$

Exercise 12. Let $h \in \mathbb{R}$ and $f \in L^2(\mathbb{R}_+)$. Prove that

$$R^*(\Theta_h^+ f) = H_h R^*(f). \tag{1}$$

Exercise 13. Let $S \in \mathcal{L}(\mathcal{A}^2(\Pi))$. Prove that the following conditions are equivalent:

(a) S is invariant under horizontal shifts:

$$\forall h \in \mathbb{R} \qquad SH_h = H_h S.$$

(b) RSR^* is invariant under multiplication by Θ_h^+ for all $h \in \mathbb{R}$:

$$\forall h \in \mathbb{R} \qquad RSR^* M_{\Theta_h^+} = M_{\Theta_h^+} RSR^*.$$

(c) There exists a function $\sigma \in L^\infty(\mathbb{R}_+)$ such that

$$S = R^* M_\sigma R.$$

(d) The Berezin transform of S is a vertical function:

$$\forall z \in \Pi \qquad \mathcal{B}(S)(z) = \mathcal{B}(S)(\Im(z)).$$