A special approximate unit

Objectives. Study some basic properties of the functions ϕ_n and their Laplace transforms ψ_n. Here ϕ_n is defined on $\mathbb{R}_+$ by

$$\phi_n(v) := \frac{1}{\big((n-1)!\big)^2} \frac{d^{n-1}}{dv^{n-1}} \left(\mathrm{e}^{-v}\ v^{2n-1}\right).$$

Requirements. Gamma and beta functions and their basic properties, basic integration tecnics (change of variables and integration by parts), associated Laguerre polynomials, Laplace transform of the function $\left(\mathrm{e}^{-t}\ t^m\right)^{(k)}$.

Definition of ϕ_n

Denote $(0, +\infty)$ by $\mathbb{R}_+$.

Definition 1. For all $n \in \{1, 2, \ldots\}$ define the function ϕ_n on $\mathbb{R}_+$ by

$$\phi_n(v) := \frac{1}{\big((n-1)!\big)^2} \frac{d^{n-1}}{dv^{n-1}} \left(\mathrm{e}^{-v}\ v^{2n-1}\right). \tag{1}$$

Exercise 1. Calculate the first derivative of $\mathrm{e}^{-v}\ v^n$:

$$\frac{d}{dv}\left(\mathrm{e}^{-v}\ v^n\right) =$$

Exercise 2. Calculate $\phi_1(v)$.

Exercise 3. Calculate $\phi_2(v)$. Factorize e^{-v} and the maximal possible power of v.

Exercise 4. Calculate $\phi_3(v)$. Factorize e^{-v} and the maximal possible power of v.

Functions ϕ_n and associated Laguerre polynomials

Exercise 5. Find the Rodrigues representation of the **associated Laguerre polynomials** (= generalized Laguerre polynomials = Sonine polynomials):

$$L_m^k(v) = \qquad \frac{d^m}{dv^m}\left(\mathrm{e}^{-v} \qquad \right). \tag{2}$$

Find also the explicit formula for the associated Laguerre polynomials:

$$L_m^k(v) = \sum \tag{3}$$

Exercise 6. Comparing the definition (1) of ϕ_n with (2) express ϕ_n through some associated Laguerre polynomial.

$$\phi_n(v) = \tag{4}$$

Exercise 7. Calculate the limits:

$$\lim_{v\to 0^+} \phi_n(v) =$$

$$\lim_{v\to +\infty} \phi_n(v) =$$

Exercise 8. Prove that ϕ_n is bounded on $\mathbb{R}_+$.

Some basic properties of the gamma function (review)

Exercise 9. Recall the definition of the gamma function:

$$\Gamma(x) \coloneqq \int\limits_0^{+\infty} \underbrace{\qquad\qquad\qquad}_{?} \, dt.$$

Exercise 10. Integrating by parts express $\Gamma(x+1)$ through $\Gamma(x)$:

$$\Gamma(x+1) =$$

Exercise 11. Compute $\Gamma(1)$:

$$\Gamma(1) = \int\limits_0^{+\infty}$$

Exercise 12. Express $\Gamma(n)$ through the factorial function for $n \in \{1, 2, 3, \ldots\}$.

$$\Gamma(n) =$$

Exercise 13. Let $a > 0$ and $p > 0$. Express the following integral through the gamma function (make a suitable change of variables):

$$\int\limits_0^{+\infty} x^p \, \mathrm{e}^{-ax} \, dx =$$

Some basic propierties of the beta function (review)

Exercise 14. Recall the definition of the beta function:

$$\mathrm{B}(x,y) := \int_0^1 \underbrace{\qquad\qquad\qquad\qquad}_{?} \, du.$$

Exercise 15. Recall the formula that expresses the beta function through the gamma function:

$$\mathrm{B}(x,y) = \underline{\qquad\qquad\qquad\qquad}.$$

Exercise 16. Let $p, q \in \{1, 2, 3, \ldots\}$. Using the formula from the previous exercise express $\mathrm{B}(p,q)$ through some factorials.

$$\mathrm{B}(p,q) =$$

Exercise 17. Using a suitable change of variables write $B(x,y)$ as an integral of the following form:

$$\mathrm{B}(x,y) = \int_0^{+\infty} \frac{t^{?}}{(1+t)^{?}} \, dt.$$

Exercise 18. Express the following integral through the beta function:

$$\int_0^{+\infty} \frac{t^a \, dt}{(1+t)^b} = \mathrm{B}(\underbrace{\qquad\qquad}_{?}, \underbrace{\qquad\qquad}_{?}).$$

Laplace transform of the function $\left(\mathrm{e}^{-t}\, t^m\right)^{(k)}$ (review)

Exercise 19. Recall the definition of the Laplace transform $\mathcal{L}(f)$ of a function f:

$$(\mathcal{L}(f))(s) \coloneqq \int \qquad\qquad\qquad dt.$$

Exercise 20. Calculate the Laplace transform of the function $\mathrm{e}^{-t}\, t^m$:

$$\int \mathrm{e}^{-t}\, t^m \qquad\qquad dt =$$

Exercise 21. Put $h(t) \coloneqq \mathrm{e}^{-t}\, t^m$. Let $k \in \{0, 1, \ldots, m-1\}$ Express the $h^{(k)}$ through a certain associated Laguerre polynomial.

Exercise 22. Let $k \in \{0, 1, \ldots, m-1\}$. Calculate the limits:

$$\lim_{s\to 0^+} h^{(k)}(s) = \qquad\qquad\qquad\qquad \lim_{s\to +\infty} h^{(k)}(s) =$$

Exercise 23. Let $k \in \{0, 1, \ldots, m-1\}$. Calculate the Laplace transform of $h^{(k)}$:

$$\int \left(\mathrm{e}^{-t}\, t^m\right)^{(k)} \qquad\qquad dt =$$

ψ_n := the Laplace transform of ϕ_n

Recall the definition of ϕ_n:

$$\phi_n(v) := \frac{1}{\big((n-1)!\big)^2} \frac{d^{n-1}}{dv^{n-1}} \left(\mathrm{e}^{-v}\ v^{2n-1}\right).$$

Definition 2. For each $n \in \{1, 2, \ldots\}$ define the function $\psi_n \colon \mathbb{R}_+ \to \mathbb{R}$ as the Laplace transform of the function ϕ_n:

$$\psi_n(t) := \int\limits_0^{+\infty} \phi_n(v)\ \mathrm{e}^{-vt}\ dv. \tag{5}$$

Exercise 24. Calculate ψ_1.

Exercise 25. Calculate ψ_2.

Exercise 26. Calculate ψ_3.

Exercise 27. Calculate $\psi_n(t)$ using the result of the Exercise 23. The coefficient in the answer can be written through the beta function.

Exercise 28. Write ψ_1, ψ_2 and ψ_3 using the result of the Exercise 23. Compare with the results of the Exercises 24, 25, 26.

Some properties of ψ_n

Exercise 29. Let $n \in \{1, 2, 3, \ldots\}$. Calculate the integral of ψ_n on $\mathbb{R}_+$:

$$\int_0^{+\infty} \psi_n(t)\, dt =$$

Exercise 30. To verify the result of the previous exercise, calculate the following integral:

$$\int_0^{+\infty} \psi_1(t)\, dt =$$

Exercise 31. Let $\delta > 0$. Prove that

$$\lim_{n\to\infty} \sup_{0<t\leq \mathrm{e}^{-\delta}} \psi_n(t) = 0.$$

Exercise 32. Let $\delta > 0$. Calculate the limit:

$$\lim_{n\to\infty} \int_0^{\mathrm{e}^{-\delta}} \psi_n(t)\, dt =$$

Exercise 33. Let $\delta > 0$. Make the change of variables $s = \dfrac{1}{t}$ in the following integral:

$$\int_{\mathrm{e}^{\delta}}^{+\infty} \psi_n(t)\, dt =$$

Exercise 34. Let $\delta > 0$. Calculate the limit:

$$\lim_{n\to\infty} \int_{(0,\mathrm{e}^{-\delta})\cup(\mathrm{e}^{\delta},+\infty)} \psi_n(t)\, dt =$$