Some basic properties of the gamma and beta functions (review)

Objectives. Recall the definition of the gamma and beta functions. Express some integrals through the gamma and beta functions. For $m, n \in \{1, 2, \ldots\}$, express $\Gamma(n)$ and $\mathrm{B}(m, n)$ through some factorials.

Requirements. Gamma function and its basic properties, beta function and its basic properties, expression of the beta function through the gamma function, basic integration tecnics (change of variables and integration by parts).

Definition of the gamma function through the Euler integral of the second kind

Exercise 1. Recall the definition of the gamma function:

$$\Gamma(x) := \int\limits_0^{+\infty} \underbrace{\qquad\qquad\qquad}_{?} \, dt.$$

Although this definition works for some complex x, we suppose that x is real and $x > \underbrace{\quad}_{?}$.

Exercise 2. Let $p > -1$. Express the following integral through the gamma function:

$$\int\limits_0^{+\infty} t^p \, \mathrm{e}^{-t} \, dt = \underbrace{\qquad\qquad}_{?}.$$

Exercise 3. Let $a > 0$ and $p > -1$. Express the following integral through the gamma function (make a suitable change of variables):

$$\int\limits_0^{+\infty} x^p \, \mathrm{e}^{-ax} \, dx =$$

Recurrence formula for the gamma function

Exercise 4. Let $p \in \mathbb{R}$. Calculate the limit:

$$\lim_{t \to +\infty} \left(t^p \, \mathrm{e}^{-t} \right) =$$

Exercise 5. Calculate the derivative of t^x with respect to t:

$$\frac{d}{dt} \left(t^x \right) =$$

Exercise 6. Let $x > 0$. Integrating by parts express $\Gamma(x+1)$ through $\Gamma(x)$:

$$\Gamma(x+1) =$$

Gamma function and factorials

Exercise 7. Compute $\Gamma(1)$:

$$\Gamma(1) = \int_0^{+\infty}$$

Exercise 8. Compute $\Gamma(2)$, $\Gamma(3)$, $\Gamma(4)$, $\Gamma(5)$:

$$\Gamma(2) = \underbrace{\qquad}_{?} \Gamma(1) = \qquad\qquad \Gamma(3) = \underbrace{\qquad}_{?} \Gamma(2) =$$

$$\Gamma(4) = \qquad\qquad \Gamma(5) =$$

Exercise 9. Let $n \in \{1, 2, 3, \ldots\}$. Express $\Gamma(n)$ through the factorial function.

$$\Gamma(n) =$$

Definition of the beta function

Exercise 10. Recall the definition of the beta function:

$$\mathrm{B}(x,y) := \int\limits_0^1 \underbrace{\qquad\qquad\qquad\qquad}_{?}\, du.$$

We suppose that x, y are reals and $x, y > \underbrace{\quad}_{?}$.

Exercise 11 (the beta function is symmetric)**.** Using the change of variables $w = 1 - u$ prove that $\mathrm{B}(y,x) = \mathrm{B}(x,y)$.

$$\mathrm{B}(y,x) =$$

Exercise 12. Express the following integral through the beta function:

$$\int\limits_0^1 u^\alpha (1-u)^\beta \, du = \mathrm{B}(\underbrace{\qquad\qquad\qquad}_{?}).$$

Exercise 13. Recall the formula that expresses the beta function through the gamma function:

$$\mathrm{B}(x,y) = \frac{\qquad\qquad}{\qquad\qquad}.$$

Exercise 14. Let $p, q \in \{1, 2, 3, \ldots\}$. Using the formula from the previous exercise express $\mathrm{B}(p,q)$ through some factorials.

$$\mathrm{B}(p,q) =$$

Beta function as a certain integral over $(0, +\infty)$

Exercise 15. Prove that the function $u \mapsto \dfrac{u}{1-u}$ is strictly increasing on $(0, 1)$.

Exercise 16. Calculate the limits:

$$\lim_{u \to 0^+} \frac{u}{1-u} = \underbrace{\qquad\qquad}_{?}, \qquad\qquad \lim_{u \to 1^-} \frac{u}{1-u} = \underbrace{\qquad\qquad}_{?}.$$

Exercise 17. Let $u \in (0, 1)$ and $t = \dfrac{u}{1-u}$. Express u through t.

Exercise 18. Making the change of variables $t = \dfrac{u}{1-u}$ write the beta function as a certain integral from 0 to $+\infty$.

$$\mathrm{B}(x, y) = \int_0^1 \underbrace{\qquad\qquad\qquad\qquad\qquad}_{?} du =$$

The answer will be of the form

$$\int_0^{+\infty} \frac{t^?}{(1+t)^?}\, dt.$$

Exercise 19. Prove that the function $u \mapsto \dfrac{1-u}{u}$ is strictly decreasing on $(0,1)$.

Exercise 20. Calculate the limits:

$$\lim_{u\to 0^+} \frac{1-u}{u} = \underbrace{\qquad}_{?}, \qquad\qquad \lim_{u\to 1^-} \frac{1-u}{u} = \underbrace{\qquad}_{?}.$$

Exercise 21. Let $u \in (0,1)$ and $t = \dfrac{1-u}{u}$. Express u through t.

Exercise 22. Making the change of variables $t = \dfrac{1-u}{u}$ write the beta function as a certain integral from 0 to $+\infty$.

Exercise 23. Compare the results of the Exercises 18 and 22.

Exercise 24. Express the following integral through the beta function:

$$\int_0^{+\infty} \frac{t^a}{(1+t)^b}\, dt =$$