Orthogonality of the monomials with respect to the weighted measure on the unit disk

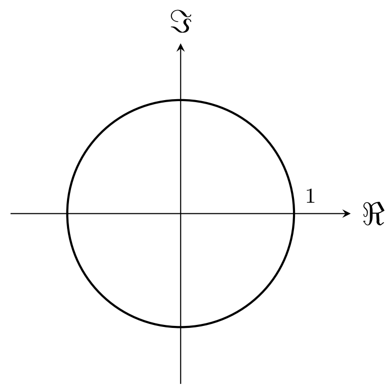
Definition 1 (weighted measure on the unit disk)**.** Denote by $dv = dx\,dy$ the standard Lebesgue plane measure, and by μ_α the following weighted measure:

$$d\mu_\alpha(z) = \frac{\alpha+1}{\pi}(1-|z|^2)^\alpha\,dv(z).$$

Objectives. Prove that $\mu_\alpha(\mathbb{D}) = 1$ and calculate the inner products of the monomial functions z^n in the space $L^2(\mathbb{D}, d\mu_\alpha)$.

Requirements. Exponential of complex arguments, change of variables in area integral, polar change of variables, beta function.

Unit circle in the complex plane



Exercise 1. Let $\varphi \in \mathbb{R}$. Using the Euler's formula express $e^{i\varphi}$ through $\cos(\varphi)$ and $\sin(\varphi)$:

$$e^{i\varphi} =$$

Exercise 2. Recall the geometrical meaning of $e^{i\varphi}$ (draw in the picture).

Exercise 3. Let $\varphi \in \mathbb{R}$. Recall the formula: $\overline{e^{i\varphi}} =$

Exercise 4. Let $k \in \mathbb{Z}$. Calculate: $e^{2k\pi i} =$

Polar coordinates

Exercise 5. Consider the polar change of variables:

$$\begin{bmatrix} x(r,\varphi) \\ y(r,\varphi) \end{bmatrix} = \begin{bmatrix} r\cos(\varphi) \\ r\sin(\varphi) \end{bmatrix}.$$

Calculate the Jacobian of the polar change of variables:

$$\begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \varphi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \varphi} \end{vmatrix} =$$

Exercise 6. Passing to the polar coordinates one must substitute $dv(z) = dx\,dy$ by

$$\underbrace{\qquad\qquad}_{?}.$$

Exercise 7. Let $z = x + \mathrm{i}\,y$. Write the following expressions in terms of the polar coordinates:

$$z = \qquad\qquad \overline{z} = \qquad\qquad |z| = \qquad\qquad |z|^2 =$$

Exercise 8. Write the weighted measure in the explicit manner and pass to the polar coordinates in the following integral (since f is a general function, it is not possible to calculate or simplify very much the integral):

$$\int_{\mathbb{D}} f(z)\,d\mu_\alpha(z) =$$

Exercise 9. Prove that $\mu_\alpha(\mathbb{D}) = 1$:

$$\mu_\alpha(\mathbb{D}) = \int_{\mathbb{D}} d\mu_\alpha(z) =$$

Orthonormal Fourier basis in $L^2\left([0,2\pi],\frac{dx}{2\pi}\right)$

Definition 2. For each $k\in\mathbb{Z}$, denote by $f_k\colon [0,2\pi]\to\mathbb{C}$ the function defined by:

$$f_k(\varphi) := \mathrm{e}^{k\,\mathrm{i}\,\varphi}.$$

Exercise 10. Let $k\in\mathbb{Z}$. Recall the formula for the derivative of f_k:

$$f_k'(\varphi) =$$

Exercise 11. Let $k\in\mathbb{Z}\setminus\{0\}$. Find an antiderivative of f_k:

$$\Bigl(\underbrace{\qquad\qquad\qquad}_{?}\Bigr)' = \mathrm{e}^{k\,\mathrm{i}\,\varphi}.$$

Exercise 12. Let $k\in\mathbb{Z}\setminus\{0\}$. Calculate the integral:

$$\frac{1}{2\pi}\int_0^{2\pi} f_k(\varphi)\,d\varphi =$$

Exercise 13. Let $k=0$. Calculate the integral:

$$\frac{1}{2\pi}\int_0^{2\pi} f_k(\varphi)\,d\varphi =$$

Exercise 14. Let $k\in\mathbb{Z}$. Write a general formula:

$$\frac{1}{2\pi}\int_0^{2\pi} f_k(\varphi)\,d\varphi =$$

Exercise 15. Let $m,n\in\mathbb{Z}$. Calculate the integral:

$$\frac{1}{2\pi}\int_0^{2\pi} f_m(x)\overline{f_n(x)}\,dx =$$

Orthogonality of the monomials in $L^2(\mathbb{D}, d\mu_\alpha)$

Denote the set $\{0, 1, 2, \ldots\}$ by $\mathbb{N}_0$.

Exercise 16. Let $m, n \in \mathbb{N}_0$, $m \neq n$. Calculate the integral:

$$\int_{\mathbb{D}} z^m \overline{z^n} \, d\mu_\alpha(z) =$$

Definition of the beta function

Exercise 17. Recall the definition of the beta function:

$$\mathrm{B}(x, y) := \int_0^1 \underbrace{\qquad\qquad\qquad}_{?} \, dt.$$

Exercise 18. Express the following integral in terms of the beta function:

$$\int_0^1 t^\alpha \, (1 - t)^\beta \, dt = \underbrace{\qquad\qquad}_{?}.$$

Norms of the monomials in $L^2(\mathbb{D}, d\mu_\alpha)$

Exercise 19. Express the following integral in terms of the beta function:

$$\int_0^1 (1 - r^2)^\alpha \, r^{2\beta} \, d \, dr =$$

Exercise 20. Let $n \in \mathbb{N}_0$. Calculate the integral:

$$\int_{\mathbb{D}} z^n \overline{z^n} \, d\mu_\alpha(z) =$$