Intertibility and spectrum of the multiplication operator on the space of square-summable sequences

Objectives. Establish an invertibility criterion and calculate the spectrum of the multiplication operator on the space ℓ^2.

Requirements. Space ℓ^2, canonical basis of ℓ^2, multiplication operator on ℓ^2 and its norm, algebraic operations (sum, product by a scalar and product) of two multiplication operators on ℓ^2, closure of a set in a metric space, bounded linear operators, norm of a bounded linear operator, invertibility of diagonal matrices, invertibility of a bounded linear operator, spectrum of a bounded linear operator.

Denote by $\mathbb{N}_0$ the set of the natural numbers starting with zero:

$$\mathbb{N}_0 \coloneqq \{0, 1, 2, \ldots\}.$$

Definition 1 (the space of the square-summable sequences of complex numbers)**.**

$$\ell^2 \coloneqq \ell^2(\mathbb{N}_0) \coloneqq \left\{ x \in \mathbb{C}^{\mathbb{N}_0} : \underbrace{\sum_j \quad x_j \qquad\qquad}_{?} \right\}.$$

The space ℓ^2 is a Hilbert space with respect to the inner product

$$\langle x, y \rangle \coloneqq \sum_{j \in \mathbb{N}_0} \overline{x_j}\, y_j.$$

This inner product is linear with respect to the second argument. Many authors define the inner product to be linear with respect to the first argument.

Definition 2 (the space of the bounded sequences of complex numbers)**.**

$$\ell^\infty \coloneqq \ell^\infty(\mathbb{N}_0) \coloneqq \left\{ x \in \mathbb{C}^{\mathbb{N}_0} : \underbrace{\qquad\qquad\qquad\qquad}_{?} \right\}.$$

Canonical basis of ℓ^2 (review)

Definition 3 (canonical basis of ℓ^2)**.** For every $n \in \mathbb{N}_0$, denote by e_n the sequence

$$e_n := \big(\delta_{n,j}\big)_{j\in\mathbb{N}_0}.$$

In the following exercises we shall see that $(e_n)_{n\in\mathbb{N}_0}$ is an orthonormal basis of ℓ^2.

Exercise 1. Write the sequence e_2: $\quad e_2 = (\underbrace{\qquad}_{?}, \underbrace{\qquad}_{?}, \underbrace{\qquad}_{?}, \underbrace{\qquad}_{?}, \underbrace{\qquad}_{?}, \ldots).$

Exercise 2. Write the following sequences:

$$-3e_0 = (\qquad, \qquad, \qquad, \qquad, \qquad, \ldots),$$

$$7e_1 = (\qquad, \qquad, \qquad, \qquad, \qquad, \ldots),$$

$$4e_0 - 5e_3 = (\qquad, \qquad, \qquad, \qquad, \qquad, \ldots).$$

Exercise 3. Write the following sequences in terms of the basic elements e_n:

$$\big(0, 0, 0, 5, 0, \ldots\big) = \underbrace{\qquad\qquad}_{?}, \qquad\qquad \big(2, 0, -4, 0, 0, \ldots\big) = \underbrace{\qquad\qquad}_{?}.$$

Exercise 4 (orthonormality of the canonical basis)**.** Let $m, n \in \mathbb{N}_0$. Calculate the product:

$$\langle e_m, e_n\rangle =$$

Exercise 5 (norm of an element of the canonical basis)**.** Let $n \in \mathbb{N}_0$. Calculate the norm of e_n:

$$\|e_n\|_2 = \underbrace{\qquad\qquad}_{?}.$$

Exercise 6 (norm of a multiple of an element of the canonical basis)**.** Let $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. Calculate the norm of λe_n:

$$\|\lambda e_n\|_2 = \underbrace{\qquad\qquad}_{?}.$$

Exercise 7. Let $x \in \ell^2$ and $n \in \mathbb{N}_0$. Calculate:

$$\langle e_n, x\rangle =$$

Exercise 8. Suppose that $x \in \ell^2$ and $\langle e_n, x\rangle = 0$ for all $n \in \mathbb{N}_0$. Then

$$\underbrace{\qquad\qquad\qquad}_{?}$$

According to general criteria of orthonormal basis in a Hilbert space, it implies that the sequence $(e_n)_{n\in\mathbb{N}_0}$ is an orthonormal basis of ℓ^2.

Exercise 9. Let $x \in \ell^2$. Write the decomposition of x in the basis $(e_n)_{n\in\mathbb{N}_0}$:

$$x = \sum_{n\in\mathbb{N}_0} \underbrace{\qquad\qquad}_{?} e_n.$$

Exercise 10. Let $x \in \ell^2$ and $m \in \mathbb{N}_0$ such that

$$\forall n \in \mathbb{N}_0 \setminus \{m\} \qquad \langle e_n, x\rangle = 0.$$

What can you say about x? Write x in terms of some basic elements e_n and some inner products.

Definition of the multiplication operator (review)

Exercise 11 (component-wise product of sequences, review)**.** Let $x, y \in \mathbb{C}^{\mathbb{N}_0}$. Write the definition of the *component-wise product* of the sequences x and y:

$$\forall n \in \mathbb{N}_0 \qquad (xy)_n \coloneqq \underbrace{\qquad\qquad}_{?}.$$

In other words,

$$xy \coloneqq \big(\underbrace{\qquad\qquad}_{?}\big)_{n\in\mathbb{N}_0}.$$

Exercise 12 (definition of the multiplication operator, review)**.** Let $a \in \ell^\infty$. Define $M_a \colon \ell^2 \to \ell^2$ by

$$M_a x \coloneqq \underbrace{\qquad\qquad}_{?},$$

that is,

$$\forall x \in \ell^2 \qquad \forall j \in \mathbb{N}_0 \qquad (M_a x)_j \coloneqq \underbrace{\qquad\qquad}_{?}.$$

Exercise 13 (action of the multiplication operator on the canonical basis, review)**.** Let $a \in \ell^\infty$ and $n \in \mathbb{N}_0$. Calculate $M_a e_n$.

Exercise 14. Let $a \in \ell^\infty$ and $n \in \mathbb{N}_0$. Calcule the ℓ^2-norm of the sequence $M_a e_n$:

$$\|M_a e_n\|_2 = \underbrace{\qquad\qquad}_{?}.$$

Supremum and infimum (review)

Definition 4. Denote by $\overline{\mathbb{R}}$ the *extended real line* $\mathbb{R} \cup \{-\infty, +\infty\}$ with the canonical order. The additional elements $-\infty$ and $+\infty$ satisfy

$$\forall \alpha \in \mathbb{R} \qquad -\infty < a, \qquad\qquad \forall \alpha \in \mathbb{R} \qquad \alpha < +\infty, \qquad\qquad -\infty < +\infty.$$

Exercise 15 (upper bound of a set of real numbers)**.** Let $A \subset \overline{\mathbb{R}}$ and $\beta \in \overline{\mathbb{R}}$. Recall the definition:

$$\beta \quad \text{is an upper bound of} \quad A \qquad \Longleftrightarrow \qquad \underbrace{\hspace{8em}}_{?}$$

Exercise 16. Let $A \subset \overline{\mathbb{R}}$ and $\gamma \in \overline{\mathbb{R}}$. Then,

$$\gamma \quad \text{is not an upper bound of} \quad A \qquad \Longleftrightarrow \qquad \underbrace{\hspace{8em}}_{?}$$

Exercise 17. Let $A \subset \overline{\mathbb{R}}$ and $\beta \in \overline{\mathbb{R}}$. What does mean the phrase "β is the supremum of A"? Write the definition using the concept of the upper bounds. (It is known that for every $A \subset \overline{\mathbb{R}}$ there exists a unique supremum in $\overline{\mathbb{R}}$. We shall not prove this fact here.)

$$\beta = \sup(A) \qquad \Longleftrightarrow$$

Exercise 18. Let $A \subset \overline{\mathbb{R}}$ and $\beta \in \overline{\mathbb{R}}$. What does mean the phrase "β is the supremum of A"? Write the answer in terms of the quantifications $\forall$ and $\exists$ and some of the inequalities $<, >, \le, \ge$, without mentioning explicitly the concept of upper bounds.

$$\beta = \sup(A) \qquad \Longleftrightarrow \qquad \begin{cases} 1) \quad \forall \alpha \in A \quad \ldots \\ \\ 2) \quad \forall \gamma < \beta \quad \ldots \end{cases}$$

Exercise 19. Let $x = (x_n)_{n \in \mathbb{N}_0}$ be a sequence in $\mathbb{R}$ and $\beta \in \overline{\mathbb{R}}$. What does mean the phrase "β is the supremum of the sequence x"? Write the answer in terms of the quantifications $\forall$ and $\exists$ and some of the inequalities $<, >, \le, \ge$.

$$\beta = \sup_{n \in \mathbb{N}_0} x_n \qquad \Longleftrightarrow \qquad \begin{cases} 1) \quad \forall n \in \mathbb{N}_0 \quad \ldots \\ \\ 2) \end{cases}$$

Multiplication operator:
boundness and norm (review)

Exercise 20 (definition of the supremum norm, review)**.** Let $a \in \ell^\infty$. Put

$$\nu := \|a\|_\infty = \sup_{n\in\mathbb{N}_0} |a_n|.$$

Write the definition of ν using the quantifications $\forall$ and $\exists$ and some of the inequalities $<, >, \leq, \geq$, without mentioning explicitly the concept of upper bounds.

Exercise 21 (idea of the upper bound for the norm of the multiplication operator, review)**.** Let $a \in \ell^\infty$ and $x \in \ell^2$. Write an upper bound for the norm of $M_a x$.

$$\|M_a x\|_2 \leq \underbrace{\qquad\qquad\qquad}_{?}.$$

Exercise 22 (idea of the lower bound for the norm of the multiplication operator, review)**.** Let $a \in \ell^\infty$. Compare $\|M_a\|$ to $\|M_a e_n\|_2$.

Exercise 23 (norm of the multiplication operator, review)**.** Let $a \in \ell^\infty$. Then

$$\|M_a\| = \underbrace{\qquad\quad}_{?}.$$

Algebraic operations with multiplication operators, review

In the following exercises recall that the set of the multiplication operators is closed under algebraic operations: for example, the sum of two multiplication operators results to be also a multiplication operator.

Exercise 24 (sum of two multiplication operators, review)**.** Let $a, b \in \ell^\infty$. Then

$$M_a + M_b = \underbrace{\qquad\qquad}_{?}.$$

Exercise 25 (product of a multiplication operator by a scalar, review)**.** Let $a \in \ell^\infty$ and $\lambda \in \mathbb{C}$. Then

$$\lambda M_a = \underbrace{\qquad\qquad}_{?}.$$

Exercise 26 (product of two multiplication operators)**.** Let $a, b \in \ell^\infty$. Then

$$M_a M_b = \underbrace{\qquad\qquad}_{?}.$$

Exercise 27 (adjoint to the multiplication operator)**.** Let $a \in \ell^\infty$. Then

$$M_a^* = \underbrace{\qquad\qquad}_{?}.$$

Exercise 28 (identity operator as a multiplication operator)**.** Denote by I the identity operator on the space ℓ^2. Find a function a such that $M_a = I$.

Range and closed range of a sequence

Exercise 29 (definition of the closure of a set in a metric space, review)**.** Let (X, d) be a metric space, $A \subset X$ and $\beta \in X$. What does mean the phrase "β is a *point of closure* of A" (in other words, "β is an *adherent point* of A")? Write the definition using quantifications and inequalities:

$$b \in \operatorname{clos}(A) \qquad \Longleftrightarrow$$

Definition 5 (range and closed range of a sequence of complex numbers)**.** Let $a \colon \mathbb{N}_0 \to \mathbb{C}$. Denote by $\mathcal{R}(a)$ the range (the set of the values) of a:

$$\mathcal{R}(a) := \big\{ z \in \mathbb{C} \colon \quad \exists n \in \mathbb{N}_0 \quad z = a_n \big\}$$

and by $\mathcal{CR}(a)$ the closure of the range of a:

$$\mathcal{CR}(a) := \operatorname{clos}(\mathcal{R}(a)).$$

Exercise 30. Let $a \colon \mathbb{N}_0 \to \mathbb{C}$. Write the definition of $\mathcal{CR}(a)$ using quantifications and inequalities:

$$w \in \mathcal{CR}(a) \qquad \Longleftrightarrow$$

Exercise 31 (example)**.** Consider a sequence $a \colon \mathbb{N}_0 \to \mathbb{C}$ is defined by

$$a_n = \big(1 + (-1)^n\big) \frac{n}{n+1}.$$

Calculate some first components of a.

$$a = \Bigg(\qquad , \qquad , \qquad , \qquad , \qquad , \ \ldots \Bigg).$$

Find $\mathcal{R}(a)$ and $\mathcal{CR}(a)$.

Injectivity and kernel of a linear operator (review)

Exercise 32 (definition of injective function)**.** Let X, Y be some sets and $f\colon X \to Y$ be a function. What does mean the phrase "f is injective"?

Exercise 33 (definition of the kernel of a linear operator)**.** Let $T\colon \ell^2 \to \ell^2$ be a linear operator. Then the *kernel* (called also the *null-space*) of T is defined by:

$$\ker(T) := \big\{x \in \ell^2 \colon \underbrace{\qquad\qquad\qquad}_{?}\big\}.$$

Exercise 34 (criterium of the injectivity of linear operator)**.** Let $T\colon \ell^2 \to \ell^2$ be a linear operator. Prove that

$$T \text{ is injective} \qquad \Longleftrightarrow \qquad \ker(T) = \{0\}.$$

Exercise 35 (the kernel of a bounded linear operator is closed)**.** Let $T\colon \ell^2 \to \ell^2$ be a bounded linear operator. Prove that its kernel $\ker(T)$ is a closed set (in the sense of the norm in ℓ^2).

Invertibility of a bounded linear operator (review)

Exercise 36. Let $T\colon \ell^2 \to \ell^2$ be a bounded linear operator. It is said that T is *invertible* if there exists a bounded linear operator $S\colon \ell^2 \to \ell^2$ such that

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{?}$$

It is known that if there exists such operator S, then it is unique. In the case of existence, S is called the *inverse operator* to T and is denoted by T^{-1}.

Exercise 37. Let $T\colon \ell^2 \to \ell^2$ be an invertible bounded linear operator. Prove that T is injective.

Exercise 38. Let $T\colon \ell^2 \to \ell^2$ be an invertible bounded linear operator. Prove that for all $x \in \ell^2$,

$$\|Tx\|_2 \geq \|T^{-1}\|^{-1}\|x\|_2.$$

Exercise 39. Let $T\colon \ell^2 \to \ell^2$ be a bounded linear operator. Suppose that for all $\varepsilon > 0$ there exists a sequence $x \in \ell^2$ such that $\|x\|_2 = 1$ and $\|Tx\|_2 < \varepsilon$. Prove that T is not invertible.

Invertibility of diagonal matrices (review)

The following exercises help to review some ideas useful to study the invertibility of the multiplication operator.

Exercise 40 (diagonal matrix as a multiplication operator, review)**.** Let $a \in \mathbb{C}^3$. Denote the components of a by a_0, a_1, a_2. Denote by $\operatorname{diag}(a)$ the diagonal matrix with diagonal entries a_0, a_1, a_2:

$$\operatorname{diag}(a) := \begin{bmatrix} a_0 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_2 \end{bmatrix}.$$

Calculate the product of $\operatorname{diag}(a)$ by a general vector $x \in \mathbb{C}^3$:

$$\operatorname{diag}(a)x = \begin{bmatrix} a_0 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} =$$

Write the answer in a short form: $\quad \operatorname{diag}(a)x = \big[\underbrace{\qquad}_{?}\big]_{j=0}^{2}.$

Exercise 41 (invertibility of a diagonal matrix, review)**.** Let $a \in \mathbb{C}^3$. Recall the criterim of the invertibility of the diagonal matrix $\operatorname{diag}(a)$:

$$\operatorname{diag}(a) \text{ is invertible} \qquad \Longleftrightarrow \qquad \underbrace{\qquad\qquad\qquad\qquad}_{?}.$$

Exercise 42 (inverse to the diagonal matrix, review)**.** Let $a \in \mathbb{C}^3$ such that $\operatorname{diag}(a)$ is invertible. Write the formula for the inverse matrix:

$$\operatorname{diag}(a)^{-1} = \left[\begin{array}{c|c|c} \qquad & \qquad & \qquad \\ \hline \qquad & \qquad & \qquad \\ \hline \qquad & \qquad & \qquad \end{array}\right].$$

Using the notation $1/a = (1/a_0, 1/a_1, 1/a_2)$ write the result in a short form:

$$\operatorname{diag}(a)^{-1} = \underbrace{\qquad\qquad\qquad}_{?}.$$

Invertibility of the multiplication operator

Exercise 43. Let $a \in \ell^\infty$. Prove that the following conditions are equivalent:

$$0 \notin \mathcal{CR}(a) \qquad \Longleftrightarrow \qquad \inf_{j \in \mathbb{N}_0} |a_j| > 0.$$

Exercise 44. Let $a \in \ell^\infty$ such that $0 \notin \mathcal{CR}(a)$. Prove that M_a is invertible. (Construct the inverse operator.)

Exercise 45. Let $a \in \ell^\infty$ such that $0 \in \mathcal{R}(a)$. Prove that M_a is not injective and therefore not invertible.

Exercise 46. Let $a \in \ell^\infty$ such that $0 \in \mathcal{CR}(a)$. Prove that for all $\varepsilon > 0$ there exists a sequence $x \in \ell^2$ such that $\|x\|_2 = 1$ and

$$\|M_a x\|_2 < \varepsilon.$$

Exercise 47. Let $a \in \ell^\infty$ such that $0 \in \mathcal{CR}(a)$. Prove that M_a is not invertible.

Exercise 48. Let $a \in \ell^\infty$. Write a criterion:

$$M_a \text{ is invertible} \qquad \Longleftrightarrow$$

Spectrum of the multiplication operator

Exercise 49. Let A be a bounded linear operator in a complex Banach space X. Recall the definition of the spectrum of A.

$$\mathrm{Sp}(A) := \left\{ \lambda \in \mathbb{C} : \qquad\qquad\qquad\qquad\qquad\qquad \right\}.$$

Exercise 50. Let $a \in \ell^\infty$. Find the spectrum of M_a.

Exercise 51. Let A be a bounded linear operator in a complex Banach space X and $\lambda \in \mathbb{C}$. What does mean the condition "λ is an eigenvalue of A?

Exercise 52. Let A be a bounded linear operator in a complex Banach space X and $\lambda \in \mathbb{C}$. Recall the definition of the point spectrum of A:

$$\mathrm{Sp}_p(A) := \left\{ \lambda \in \mathbb{C} : \qquad\qquad\qquad\qquad\qquad\qquad \right\}.$$

Exercise 53. Let $a \in \ell^\infty$. Find the point spectrum of the multiplication operator M_a.

Exercise 54 (spectrum of the multiplication operator in terms of its point spectrum)**.** Let $a \in \ell^\infty$. Express $\mathrm{Sp}(M_a)$ in terms of $\mathrm{Sp}_p(M_a)$.

Multiplication operator and canonical basis

Exercise 55. Let $a \in \ell^\infty$, $x \in \ell^2$ and $n \in \mathbb{N}_0$. Calculate:

$$\langle e_n, M_a x \rangle =$$

Exercise 56. Let $a \in \ell^\infty$ and $x \in \ell^2$. Prove that

$$M_a x = \sum_{j \in \mathbb{N}_0} a_j \langle e_j, x \rangle \, e_j.$$

Exercise 57. Let $a \in \ell^\infty$ and $j \in \mathbb{N}_0$. Calculate:

$$\langle e_j, M_a e_j \rangle =$$

Exercise 58. Let $a \in \ell^\infty$ and the sequence $(\lambda_j)_{j \in \mathbb{N}_0}$ is defined by:

$$\forall j \in \mathbb{N}_0 \qquad \lambda_j \coloneqq \langle e_j, M_a e_j \rangle.$$

Express a through λ_j.

Exercise 59. Let $a \in \ell^\infty$ and $j, k \in \mathbb{N}_0$ such that $j \neq k$. Calculate:

$$\langle e_j, M_a e_k \rangle =$$