Modulus of continuity of functions with respect to the logarithmic distance

Objectives. Study some properties of the *modulus of continuity* $\Omega_{\rho,f}$,

$$\Omega_{\rho,f}(\delta) := \sup\big\{|f(x) - f(y)|: \quad x, y \in \mathbb{R}_+, \quad \rho(x,y) \le \delta\big\},$$

of a function $f\colon \mathbb{R}_+ \to \mathbb{C}$ with respect to the *logarithmic distance* $\rho\colon \mathbb{R}_+ \times \mathbb{R}_+ \to [0, +\infty)$ defined by

$$\rho(x,y) := \big|\ln(x) - \ln(y)\big|.$$

Requirements. Properties of the logarithmic distance ρ.

Logarithmic distance (review)

Exercise 1. Fill the table:

	$\max(x,y)$	$\min(x,y)$	$\big\lvert\ln(x) - \ln(y)\big\rvert$
Case $x \ge y$:			
Case $x < y$:			

Exercise 2. Express $\rho(x,y)$ through $\max(x,y)$ and $\min(x,y)$:

$$\rho(x,y) \;=\; \ln\Big(\qquad\qquad \Big) - \ln\Big(\qquad\qquad \Big) \;=\; \ln \frac{\qquad}{\qquad}.$$

Exercise 3 (ρ is invariant with respect to the dilations)**.** Let $x, y, t \in \mathbb{R}_+$. Simplify:

$$\rho(tx, ty) =$$

Exercise 4. Let $x, y \in \mathbb{R}_+$. Then

$$\rho(x,y) = \rho\Big(\qquad\quad , \; 1\Big).$$

$\boldsymbol{\delta}$-neighborhoods of 1 with respect to $\boldsymbol{\rho}$ (review)

Exercise 5 (left δ-neighborhood of 1)**.**
Let $\delta > 0$. Find all $u \in (0, 1]$ such that $\rho(u, 1) \le \delta$.

Exercise 6 (right δ-neighborhood of 1)**.**
Let $\delta > 0$. Find all $u \in [1, +\infty)$ such that $\rho(u, 1) \le \delta$.

Exercise 7 (δ-neighborhood of 1)**.**
Let $\delta > 0$. Find all $u > 0$ such that $\rho(u, 1) \le \delta$.

$\boldsymbol{\delta}$-entourages with respect to the distance $\boldsymbol{\rho}$ (review)

Exercise 8. Let $\delta > 0$ and $x \in \mathbb{R}_+$. Find all $y \in [x, +\infty)$ such that $\rho(x, y) \le \delta$.

Exercise 9. Let $\delta > 0$ and $x \in \mathbb{R}_+$. Find all $y \in (0, x]$ such that $\rho(x, y) \le \delta$.

Exercise 10. Let $\delta > 0$ and $x \in \mathbb{R}_+$. Find all $y \in \mathbb{R}_+$ such that $\rho(x, y) \le \delta$.

Definition of the modulus of continuity of a function with respect to the logarithmic distance

Definition 1 (modulus of continuity of a function with respect to the logarithmic distance)**.** Let $f\colon \mathbb{R}_+ \to \mathbb{C}$. Define $\Omega_{\rho,f}\colon \mathbb{R}_+ \to [0,+\infty]$ by

$$\Omega_{\rho,f}(\delta) \coloneqq \sup\big\{|f(x)-f(y)|:\quad x,y>0,\quad \rho(x,y)\le\delta\big\}.$$

Exercise 11. Write another description of $\Omega_{\rho,f}$ using the result of the Exercise 4:

$$\Omega_{\rho,f}(\delta) = \sup\Big\{|f(x)-f(y)|:\quad x,y>0,\quad \rho\Big(\underbrace{\qquad}_{?},1\Big)\le\delta\Big\}. \tag{1}$$

Exercise 12. Rewrite (1) using the result of the Exercise 7:

$$\Omega_{\rho,f}(\delta) = \sup\big\{|f(x)-f(y)|:\quad x,y>0,\quad \underbrace{\qquad}_{?}\le\underbrace{\qquad}_{?}\le\underbrace{\qquad}_{?}\big\}.$$

Exercise 13. Apply the description of the δ-entourages found in the Exercise 10:

$$\Omega_{\rho,f}(\delta) = \sup\big\{|f(x)-f(y)|:\quad x>0,\quad \underbrace{\qquad}_{?}\le y\le\underbrace{\qquad}_{?}\big\}.$$

Exercise 14. In other words, $\Omega_{\rho,f}$ can be written as a double supremum:

$$\Omega_{\rho,f}(\delta) = \sup_{x>0}\qquad\sup_{\le y\le}\qquad|f(x)-f(y)|.$$

Applying the symmetricity in the definition of the modulus of continuity

In the exercises of this section we fix a function $f\colon \mathbb{R}_+ \to \mathbb{C}$ and a number $\delta \in (0,1)$. Consider the sets

$$S \coloneqq \bigl\{|f(x)-f(y)|:\quad x,y>0,\quad \rho(x,y)\le\delta\bigr\},$$
$$S_1 \coloneqq \bigl\{|f(a)-f(b)|:\quad a,b>0,\quad a\le b,\quad \rho(a,b)\le\delta\bigr\},$$
$$S_2 \coloneqq \bigl\{|f(u)-f(v)|:\quad u,v>0,\quad u\ge v,\quad \rho(u,v)\le\delta\bigr\}.$$

We are going to find relations between the sets S_1, S_2 and S.

Exercise 15 (recall the definition of total order)**.**
A binary relation $\prec$ on a set X is called a *total order* (or *linear order*) on X if it has the following properties:

1. Transitivity:

$$\forall x,y\in X \qquad \bigl((x\prec y \quad \wedge \quad y\prec x) \quad \Rightarrow \quad \underbrace{\qquad\qquad}_{?}\bigr).$$

2. Antisymmetry:

$$\forall x,y\in X \qquad \bigl((x\prec y \quad \wedge \quad y\prec x) \quad \Rightarrow \quad \underbrace{\qquad\qquad}_{?}\bigr).$$

3. Totality:

$$\forall x,y\in X \qquad \bigl(x\prec y \quad \underbrace{\qquad\quad}_{?} \quad y\prec x\bigr).$$

Exercise 16. The usual non-string order $\le$ is a total order (= linear order) on $\mathbb{R}_+$.
It means that for all $x,y>0$ we have

$$x\le y \qquad \underbrace{\qquad}_{?} \qquad x\ge y.$$

Therefore S can be expressed through S_1 and S_2 in the following manner:

$$S = S_1 \underbrace{\qquad}_{?} S_2.$$

Exercise 17. The expression $|f(x) - f(y)|$ is symmetric with respect to x and y:

$$|f(x) - f(y)| = \underbrace{\qquad\qquad\qquad}_{?}.$$

The function ρ is also symmetric: $\rho(x, y) = \underbrace{\qquad\qquad}_{?}.$

Exercise 18. Let $s \in S_1$. By the definition of S_1 there exist $a, b > 0$ such that

$$s = |f(a) - f(b)|, \qquad a \underbrace{\qquad}_{?} b, \qquad \rho(a, b) \underbrace{\qquad\quad}_{?}.$$

Apply the symmetricity of $|f(a) - f(b)|$:

$$s = |f(a) - f(b)| = |f(\underbrace{\qquad}_{?}) - f(\underbrace{\qquad}_{?})|,$$

and the symmetricity of ρ:

$$\rho(\underbrace{\qquad}_{?}, \underbrace{\qquad}_{?}) = \rho(a, b) \le \underbrace{\qquad}_{?}.$$

Put $u := \underbrace{\qquad}_{?}, \quad v := \underbrace{\qquad}_{?}.$ Then by the previous formulas,

$$s = \underbrace{\qquad\qquad\qquad}_{?} \qquad \text{and} \qquad \rho(\underbrace{\qquad}_{?}, \underbrace{\qquad}_{?}) \le \underbrace{\qquad}_{?}.$$

By the definition of S_2 it means that $\underbrace{\qquad\qquad}_{?}.$

Since s is an arbitrary element of S_1, the following inclusion holds: $S_1 \underbrace{\qquad}_{?} S_2.$

It can be proved in a similar manner that $S_2 \underbrace{\qquad}_{?} S_1.$

Conclusion: $S_1 \underbrace{\qquad}_{?} S_2.$

Exercise 19. Using the results of the Exercises 16 and 18 establish a relation between the sets S, S_1, S_2.

Exercise 20. Express $\Omega_{\rho,f}(\delta)$ in terms of the set S:

$$\Omega_{\rho,f}(\delta) = \underbrace{\qquad\qquad\qquad}_{?}.$$

Taking in account the result of the Exercise 19 we can express the modulus of continuity $\Omega_{\rho,f}$ through S_1 or through S_2:

$$\Omega_{\rho,f}(\delta) = \underbrace{\qquad\qquad\qquad}_{?} = \underbrace{\qquad\qquad\qquad}_{?}. \tag{2}$$

Exercise 21. Substituting the definitions of S_1 and S_2 into (2) we obtain the following representations of the modulus of continuity $\Omega_{\rho,f}(\delta)$:

$$\Omega_{\rho,f}(\delta) \;=\; \sup\Big\{ \qquad\qquad\qquad : \quad x, y \in \underbrace{\qquad}_{?}, \qquad x \le y, \qquad \rho(x,y)\underbrace{\qquad}_{?}\Big\} \tag{3}$$

$$\Omega_{\rho,f}(\delta) \;=\; \sup\Big\{ \qquad\qquad\qquad : \quad x, y \in \underbrace{\qquad}_{?}, \qquad\qquad\qquad\qquad\qquad \Big\}. \tag{4}$$

Exercise 22. Using the results of the Exercises 8 and 9 write the formulas (3) and (3) in the following manner:

$$\Omega_{\rho,f}(\delta) \;=\; \sup\Big\{ \qquad\qquad\qquad : \quad x > 0, \qquad \underbrace{\qquad}_{?} \le y \le \underbrace{\qquad}_{?}\Big\}, \tag{5}$$

$$\Omega_{\rho,f}(\delta) \;=\; \sup\Big\{ \qquad\qquad\qquad : \qquad\qquad\qquad\qquad\qquad\qquad \Big\}. \tag{6}$$

Exercise 23. Rewrite the formulas (5) and (6) using double supremums:

$$\Omega_{\rho,f}(\delta) \;=\; \sup_{x>0} \qquad \sup_{\le y \le} \qquad |f(x) - f(y)|, \tag{7}$$

$$\Omega_{\rho,f}(\delta) \;= \tag{8}$$

The logarithmic distance and the standard distance

Definition 2 (standard distance on the real line)**.** The standard distance $d\colon \mathbb{R}\times\mathbb{R}\to\mathbb{C}$ is defined by:

$$d(t,u) \;\coloneqq\; \underbrace{\qquad\qquad}_{?}.$$

Exercise 24. Recall the definition of the logarithmic distance ρ:

$$\rho(x,y) \;\coloneqq\; \underbrace{\qquad\qquad\qquad\qquad}_{?}.$$

Exercise 25. The logarithmic distance ρ on the positive half-line can be expressed through the standard distance d on the real line in the following manner:

$$\rho(x,y) \coloneqq \underbrace{\qquad\qquad\qquad\qquad}_{?} = |t-u|,$$

there

$$t = \underbrace{\qquad\qquad}_{?}, \qquad u = \underbrace{\qquad\qquad}_{?}.$$

The variables x and y can be expressed through t and u:

$$x = \underbrace{\qquad\qquad}_{?}, \qquad y = \underbrace{\qquad\qquad}_{?}. \tag{9}$$

The function $\underbrace{\qquad\quad}_{?}$ is a bijection of $\mathbb{R}_+$ onto $\underbrace{\qquad\quad}_{?}$.

Therefore the domain of the variables t and u is the set $\underbrace{\qquad\quad}_{?}$.

Expression through the standard modulus of continuity

We suppose that $f\colon \mathbb{R}_+ \to \mathbb{C}$ and $\delta > 0$.

Exercise 26. Copy the definition of $\Omega_{\rho,f}(\delta)$:

$$\Omega_{\rho,f}(\delta) \;\coloneqq\; \sup\Big\{|f(x) - f(y)| \colon \qquad\qquad\qquad\qquad\qquad\qquad \Big\}. \tag{10}$$

Exercise 27. In the right-hand side of (10) make the change of variables (9) from the Exercise 25:

$$\Omega_{\rho,f} \;=\; \sup\Big\{\big|f(\qquad) - f(\qquad)\big| \colon \quad u, v \in \underbrace{\qquad}_{?}, \quad d(\underbrace{\qquad}_{?}, \underbrace{\qquad}_{?}) \le \underbrace{\qquad}_{?}\Big\}. \tag{11}$$

Exercise 28. The difference $\big|f(\qquad) - f(\qquad)\big|$ in the right-hand side of (11) can be written as

$$|g(t) - g(u)|, \qquad \text{where} \qquad g(t) \coloneqq \underbrace{\qquad\qquad}_{?}.$$

Exercise 29. Conclusion: $\Omega_{\rho,f}(\delta)$ can be written as $\Omega_{d,?}(?)$:

$$\Omega_{\rho,f}(\delta) \;=\; \Omega_{d,\qquad}(\qquad).$$

In other words, the δ-modulus of continuity of a function $f\colon \mathbb{R}_+ \to \mathbb{C}$
with respect to the logarithmic distance ρ
is equal to the $\underbrace{\qquad}_{?}$-modulus of continuity of the function $\underbrace{\qquad\qquad}_{?}\colon \underbrace{\qquad}_{?} \to \mathbb{C}$
with respect to the standard distance d.

Some properties of cos

The functions cos and sin are ones of the simplest bounded functions that do not have any limits at infinity. Here we recall some well-known properties of the function cos.

Exercise 30. Recall the formula for the difference of cosines:

$$\cos(t) - \cos(u) = \qquad \frac{t+u}{2} \qquad \frac{\qquad}{2}.$$

Exercise 31. Recall two standard upper bounds for $\sin(a)$, $a \in \mathbb{R}$:

$$\left|\sin(a)\right| \le \underbrace{\qquad}_{?}, \qquad \left|\sin(a)\right| \le \underbrace{\qquad}_{?}.$$

One of these upper bounds is a constant, the other is not a constant and is more precise for small values of a.

Exercise 32. Using the results of the Exercises 30 and 31 prove that cos is Lipschitz-continuous:

$$\left|\cos(t) - \cos(u)\right| \le$$

Exercise 33. Recall in what points of the real line cos takes its maximum value:

$$\cos(t) = 1 \qquad \Longleftrightarrow$$

Exercise 34. Recall in what points of the real line cos takes its minimum value:

$$\cos(u) = -1 \qquad \Longleftrightarrow$$

Example: $f(x) \coloneqq \cos(\ln(x))$

Consider the function $f\colon \mathbb{R}_+ \to \mathbb{C}$ defined by $f(x) \coloneqq \cos(\ln(x))$.

Exercise 35. Find a good upper bound for the modulus of continuity of the function f with respect to the logarithmic distance ρ:

$$\Omega_{\rho,f}(\delta) \le ?.$$

Example: $f(x) \coloneqq \cos\left(\sqrt{x}\right)$

Consider the function $f\colon \mathbb{R}_+ \to \mathbb{C}$ defined by $f(x) \coloneqq \cos\left(\sqrt{x}\right)$.

Exercise 36. Find a good lower bound for the modulus of continuity of the function f with respect to the logarithmic distance ρ:

$$\Omega_{\rho,f}(\delta) \ge ?.$$