# Logarithmic distance on the positive half-line

Denote the interval $(0, +\infty)$ by $\mathbb{R}_+$.

**Definition 1.** Define the function $\rho\colon \mathbb{R}_+ \times \mathbb{R}_+ \to [0, +\infty)$ by

$$\rho(x, y) \coloneqq \bigl|\ln(x) - \ln(y)\bigr|.$$

**Exercise 1.** Recall that for all $a, b > 0$

$$\ln(a) - \ln(b) = \ln \underline{\qquad}.$$

Therefore $\rho$ can be written in the following form:

$$\rho(x, y) =$$

**Note 1.** In the following exercise we use the functions max and min and threat them as functions of two real arguments:

$$\max(7, 3) = 7, \qquad \min(7, 3) = 3.$$

**Exercise 2.** Fill the table:

| | compare $\ln(x)$ and $\ln(y)$ | simplify $\bigl\|\ln(x) - \ln(y)\bigr\|$ | simplify $\max(x, y)$ | simplify $\min(x, y)$ |
|---|---|---|---|---|
| Case $x \geq y$: | $\ln(x) \underbrace{\qquad}_{?} \ln(y)$ | | | |
| Case $x < y$: | $\ln(x) \underbrace{\qquad}_{?} \ln(y)$ | | | |

**Exercise 3.** Express $\rho(x, y)$ through $\max(x, y)$ and $\min(x, y)$:

$$\rho(x, y) \;=\; \ln\bigl(\qquad\qquad\bigr) - \ln\bigl(\qquad\qquad\bigr) \;=\; \ln \underline{\qquad\qquad}.$$

# Direct verification that $\rho$ is a distance

We are going to prove that $\rho$ is a distance on $\mathbb{R}_+$.

**Exercise 4.** List the conditions from the definition of distance (= metric).

1.

2.

3. For all $x$, $\rho(x,x) = \underbrace{\qquad}_{?}$.

4. For all $x, y$, if $x \neq y$, then $\underbrace{\qquad\qquad\qquad}_{?}$

**Exercise 5.** Prove that $\rho$ is symmetric.

**Exercise 6.** Let $x, y \in \mathbb{R}_+$, $x \neq y$. Prove that $\rho(x,y) > 0$.

**Exercise 7.** Prove that $\rho$ fulfills the triangular inequality.

# Abstract construction: transfer of distance

**Exercise 8.** Let $(M, g)$ be a metric space, $X$ be a set and $\phi\colon X \to M$ be an injective function. Define the function $f\colon X \times X \to [0, +\infty]$ by

$$f(x, y) \coloneqq g(\phi(x), \phi(y)).$$

Prove that $f$ is a distance on $X$.

**Definition 2.** Denote by $d$ the standard distance on $\mathbb{R}$:

$$\forall t, u \in \mathbb{R} \qquad d(t, u) \coloneqq |t - u|.$$

**Exercise 9.** Explain how to apply the construction from the Exercise 8 to define the logarithmic distance $\rho$ on $\mathbb{R}_+$:

$$\rho(x, y) = \bigl|\ln(x) - \ln(y)\bigr|.$$

$X =$ $\qquad\qquad$ $M =$

$f =$ $\qquad\qquad$ $g =$

$\phi(x) =$

## $\mathbb{R}_+$ as a group

**Exercise 10.** The set $\mathbb{R}_+$ is usually considered with the binary operation $\underbrace{\qquad\qquad}_{?}$

and is a $\underbrace{\qquad\qquad\qquad\qquad}_{\text{commutative/non-commutative}}$ group.

**Exercise 11.** The function $\underbrace{\qquad\qquad}_{?}$ is an isomorphism

from the group $(\mathbb{R}, +)$ to the group $(\mathbb{R}_+, \underbrace{\qquad}_{?})$,

and the inverse isomorphism from $(\mathbb{R}_+, \underbrace{\qquad}_{?})$ to $(\mathbb{R}, +)$ is given by the function $\underbrace{\qquad\qquad}_{?}$.

## The distance $\rho$ is invariant under dilations (in other words, is homogeneous of degree 0)

**Exercise 12.** Let $x, y, t > 0$. Simplify:

$$\rho(tx, ty) =$$

**Note 2.** In other words, the Exercise 12 states that the distance $\rho$ conforms with the operation of the group $\mathbb{R}_+$.

**Exercise 13.** Let $x, y > 0$. Express $\rho(x, y)$ through $\rho\,(x/y, 1)$.

## $\boldsymbol{\delta}$-neighborhoods of 1 with respect to the distance $\boldsymbol{\rho}$

**Exercise 14** (right $\delta$-neighborhood of 1 with respect to the distance $\rho$)**.**
Let $\delta > 0$. Find all $u \in [1, +\infty)$ such that $\rho(u, 1) < \delta$.

$$\begin{cases} u \geq 1 \\ \rho(u,1) < \delta \end{cases} \quad \Longleftrightarrow \quad \begin{cases} u \geq 1 \\ \\ \end{cases} \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad \underbrace{\qquad}_{?} \leq u < \underbrace{\qquad}_{?} \quad \Longleftrightarrow \quad u \in \Big[ \qquad , \qquad \Big).$$

**Exercise 15** (left $\delta$-neighborhood of 1 with respect to the distance $\rho$)**.**
Let $\delta > 0$. Find all $u \in (0, 1]$ such that $\rho(u, 1) < \delta$.

**Exercise 16** ($\delta$-neighborhood of 1 with respect to the distance $\rho$)**.**
Let $\delta > 0$. Find all $u \in \mathbb{R}_+$ such that $\rho(u, 1) < \delta$.

## $\delta$-entourages with respect to the distance $\rho$

**Exercise 17.** Let $\delta > 0$ and $x \in \mathbb{R}_+$. Find all $y \in [x, +\infty)$ such that $\rho(x, y) < \delta$.

$$\begin{cases} y \geq x \\ \rho(x,y) < \delta \end{cases} \iff \begin{cases} \dfrac{y}{x} \\ \rho\Big(1, \dfrac{y}{x}\Big) \end{cases} \iff$$

$$\iff \underbrace{\qquad}_{?} \leq y < \underbrace{\qquad}_{?} \iff y \in \Big[ \qquad , \qquad \Big).$$

**Exercise 18.** Let $\delta > 0$ and $x \in \mathbb{R}_+$. Find all $y \in (0, x]$ such that $\rho(x, y) < \delta$.

**Exercise 19.** Let $\delta > 0$ and $x \in \mathbb{R}_+$. Find all $y \in \mathbb{R}_+$ such that $\rho(x, y) < \delta$.

## Some upper and lower bounds for the logarithmic function

**Exercise 20.** Define $f\colon [1,+\infty) \to \mathbb{R}$ by

$$f(u) \coloneqq u - 1 - \ln(u).$$

Calculate $f'(u)$ and determine the sign of $f'(u)$ for $u > 1$.
Determine if $f$ increases or decreases on $[1,+\infty)$.
Calculate $f(1)$ and determine the sign of $f(u)$ for $u > 1$.

**Exercise 21.** Let $u \geq 1$. Compare $\ln(u)$ with $u - 1$.

$$\forall u \geq 1 \qquad \ln(u) \underbrace{\qquad}_{?} u - 1.$$

**Exercise 22.** Define $f\colon [1,+\infty) \to \mathbb{R}$ by

$$f(u) \coloneqq \frac{1}{u} - 1 + \ln(u).$$

Calculate $f'(u)$ and determine the sign of $f'(u)$ for $u > 1$.
Determine if $f$ increases or decreases on $[1,+\infty)$.
Calculate $f(1)$ and determine the sign of $f(u)$ for $u > 1$.

**Exercise 23.** Let $u \geq 1$. Compare $\ln(u)$ with $1 - \frac{1}{u}$.

$$\forall u \geq 1 \qquad \ln(u) \underbrace{\qquad}_{?} 1 - \frac{1}{u}.$$

**Exercise 24.** Define $f\colon (1, +\infty) \to \mathbb{R}$ by

$$f(u) \coloneqq \frac{\ln(u)}{1 - \frac{1}{u}}.$$

Calculate $f'(u)$ and determine the sign of $f'(u)$ for $u > 1$.
Determine if $f$ increases or decreases on $[1, +\infty)$.

**Exercise 25.** Find a positive constant $C > 0$ such that $\forall t \in \left[1, \frac{3}{2}\right]$

$$\ln(t) \leq C \left(1 - \frac{1}{t}\right).$$

## Comparison to another dilation invariant distance

**Definition 3.** Define the function $\rho_1 \colon \mathbb{R}_+ \times \mathbb{R}_+ \to [0, +\infty)$ by

$$\rho_1(x, y) = \frac{|x - y|}{\max(x, y)}.$$

It can be shown that $\rho_1$ is a distance on $\mathbb{R}_+$ (we shall not prove it here).

**Exercise 26.** Express $|x - y|$ through $\max(x, y)$ and $\min(x, y)$:

$$|x - y| =$$

**Exercise 27.** Express $\rho_1(x, y)$ through $\max(x, y)$ and $\min(x, y)$:

$$\rho_1(x, y) =$$

**Exercise 28** ($\rho_1$ is dilation invariant)**.** Let $x, y, t > 0$. Simplify:

$$\rho_1(tx, ty) =$$

**Exercise 29.** Let $u \geq 1$ Recall the inequality between $\ln(u)$ and $1 - \frac{1}{u}$.

**Exercise 30.** Let $x, y \in \mathbb{R}_+$. Compare $\rho(x, y)$ with $\rho_1(x, y)$.

**Exercise 31.** Prove that for all $x, y \in \mathbb{R}_+$ such that $\rho_1(x, y) \leq \frac{1}{2}$ the distance $\rho$ can be bounded by a multiple of $\rho_1$ in the following manner:

$$\rho(x, y) \leq C\, \rho_1(x, y).$$

## Examples and counter-examples of sequences such that $\rho(x_n, x_{n+1}) \to 0$

For each one of the following sequences $(x_n)_{n=1}^{\infty}$ determine whether $\rho(x_n, x_{n+1}) \to 0$ or not.

**Exercise 32.** $x_n \coloneqq n$.

**Exercise 33.** $x_n \coloneqq n^2$.

**Exercise 34.** $x_n \coloneqq \ln(n)$.

**Exercise 35.** $x_n \coloneqq 2^n$.

**Exercise 36.** $x_n \coloneqq \dfrac{1}{2^n}$.

**Exercise 37.** $x_n \coloneqq \dfrac{1}{n}$.