A dilation-invariant distance on the positive half-line

Denote the interval $(0, +\infty)$ by $\mathbb{R}_+$.

Definition 1. Define the function $\rho_1 \colon \mathbb{R}_+ \times \mathbb{R}_+ \to [0, +\infty)$ by

$$\rho_1(x, y) \coloneqq \frac{|x - y|}{\max(x, y)}.$$

Note 1. We threat max and min as functions of two real arguments:

$$\max(7, 3) = 7, \qquad \min(7, 3) = 3.$$

One can also threat them as functions of one set argument:

$$\max\{7, 3\} = 7, \qquad \min\{7, 3\} = 3.$$

Exercise 1. Fill the table:

	$\|x - y\|$	$\max(x, y)$	$\min(x, y)$
Case $x \geq y$:			
Case $x < y$:			

Exercise 2. Express $|x - y|$ through $\max(x, y)$ and $\min(x, y)$:

$$|x - y| =$$

Exercise 3. Express $\rho_1(x, y)$ through $\max(x, y)$ and $\min(x, y)$:

$$\rho_1(x, y) = \qquad\qquad\qquad\qquad = 1 - \underline{\qquad\qquad}.$$

We are going to prove that ρ_1 is a distance on $\mathbb{R}_+$.

Exercise 4. List the conditions from the definition of distance (= metric). Check which of them are obvious for the function ρ_1.

Exercise 5. We are going to prove the triangular inequality for ρ_1. Write it for some points $x, y, z \in \mathbb{R}_+$:

$$\underbrace{\qquad\qquad}_{?} \le \underbrace{\qquad\qquad\qquad\qquad}_{?}.$$

Exercise 6. Rewrite the inequality from the previous exercise in the following form:

"something ≥ 0"

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{?} \ge 0. \qquad (1)$$

Exercise 7. Write all the possible orderings of three numbers $x, y, z \in \mathbb{R}_+$:

1) $x \le y \le z$; 2) $x \le z \le y$;

Exercise 8. The left-hand side of (1) is symmetric with respect to the following variables (choose the correct answer):

○ x and y

○ x and z

○ y and z

Exercise 9. It follows from the previous exercise that without any loss of generality we can suppose that

$$\underbrace{\qquad\qquad\qquad\qquad}_{?}$$

So, it is sufficient to consider only the following orderings of x, z, y:

1) $\quad \le \quad \le \quad$;

Exercise 10. Consider the first case from the Exercise 9. In this case x, y, z are ordered in the following manner:

$$\underbrace{\qquad}_{?} \le \underbrace{\qquad}_{?} \le \underbrace{\qquad}_{?}. \tag{2}$$

Calculate the left-hand side of the formula (1):

$$\begin{aligned} \rho_1(\qquad\quad) + \rho_1(\qquad\quad) - \rho_1(\qquad\quad) &= \\ &= \Bigg(\qquad\qquad\Bigg) + \Bigg(\qquad\qquad\Bigg) - \Bigg(\qquad\qquad\Bigg) \\ &= \\ &= \frac{(\qquad\qquad)(\qquad\qquad)}{\qquad}. \end{aligned}$$

The last expression is $\underbrace{\qquad}_{\ge \text{ or } \le} 0$ by the condition (2).

Exercise 11. Calculate the left-hand side of (1) for other cases from the Exercise 9.

The distance ρ_1 is invariant under dilations (in other words, is homogeneous of degree 0)

Exercise 12. Let $x, y, t > 0$. Simplify:

$$\rho_1(tx, ty) =$$

Exercise 13. Let $x, y > 0$. Express $\rho_1(x, y)$ through $\rho_1\left(\frac{x}{y}, 1\right)$.

Pairs of the points that are δ-close with respect to ρ_1

Exercise 14. Let $\delta \in (0, 1)$. Find all $u \in [1, +\infty)$ such that $\rho_1(u, 1) \leq \delta$.

$$\begin{cases} u \geq 1 \\ \rho_1(u, 1) \leq 1 \end{cases} \iff \begin{cases} u \geq 1 \\ \end{cases} \iff$$

Exercise 15. Let $\delta \in (0, 1)$. Find all $u \in (0, 1]$ such that $\rho_1(u, 1) \leq \delta$.

Exercise 16. Let $\delta \in (0, 1)$. Find all $u \in \mathbb{R}_+$ such that $\rho_1(u, 1) \leq \delta$.

Exercise 17. Let $\delta \in (0, 1)$ and $x \in \mathbb{R}_+$. Find all $y \in [x, +\infty)$ such that $\rho_1(x, y) \leq \delta$.

Exercise 18. Let $\delta \in (0, 1)$ and $x \in \mathbb{R}_+$. Find all $y \in (0, x]$ such that $\rho_1(x, y) \leq \delta$.

Exercise 19. Let $\delta \in (0, 1)$ and $x \in \mathbb{R}_+$. Find all $y \in \mathbb{R}_+$ such that $\rho_1(x, y) \leq \delta$.

Examples and counter-examples of sequences such that $\rho_1(x_n, x_{n+1}) \to 0$

For each one of the following sequences $(x_n)_{n=1}^{\infty}$ determine whether $\rho_1(x_n, x_{n+1}) \to 0$ or not.

Exercise 20. $x_n \coloneqq n$.

Exercise 21. $x_n \coloneqq n^2$.

Exercise 22. $x_n \coloneqq \ln(n)$.

Exercise 23. $x_n \coloneqq 2^n$.

Exercise 24. $x_n \coloneqq \dfrac{1}{2^n}$.

Exercise 25. $x_n \coloneqq \dfrac{1}{n}$.