Criteria of multiplication operator on the space of square-summable sequences

Objectives. Establish some criteria of the multiplication operator on the space ℓ^2.

Requirements. The space ℓ^2 of the square-summable sequences of complex numbers, multiplication operator by a bounded sequence on the space ℓ^2, basic properties of the multiplication operator.

Denote by $\mathbb{N}$ the set of the natural numbers. The notation $\mathbb{C}^{\mathbb{N}}$ will be used for the set of all the sequences of complex numbers, that is, for the set of all the functions $\mathbb{N} \to \mathbb{C}$.

The space of the square-summable sequences (review)

Exercise 1. Recall the definition of $\ell^2(\mathbb{N})$ and the definition of the inner product in $\ell^2(\mathbb{N})$. We suppose that the inner product is linear with respect to the second argument.

$$\ell^2 := \ell^2(\mathbb{N}) := \Bigg\{ x \in \mathbb{C}^{\mathbb{N}} \colon \ \sum \qquad\qquad\qquad\qquad \Bigg\}, \qquad\qquad \langle x, y \rangle := \sum \qquad\quad .$$

Exercise 2 (the space of the bounded sequences of complex numbers)**.**

$$\ell^\infty := \ell^\infty(\mathbb{N}) := \Bigg\{ x \in \mathbb{C}^{\mathbb{N}} \colon \qquad\qquad\qquad\qquad \Bigg\}.$$

Canonical base of ℓ^2 (review)

Exercise 3 (canonical base of ℓ^2)**.** For every $n \in \mathbb{N}$, the sequence e_n is defined by:

$$e_n := (\underbrace{\qquad\qquad}_{?})_{j\in\mathbb{N}}.$$

Exercise 4. Let $x \in \ell^2$ and $n \in \mathbb{N}$. Calculate: $\quad \langle e_n, x\rangle = \qquad\qquad .$

Exercise 5. It is known that $(e_n)_{n\in\mathbb{N}}$ is an orthonormal base of ℓ^2. Therefore every $x \in \ell^2$ can be decomposed in the following manner:

$$x = \sum_{n\in\mathbb{N}} \langle \quad , \quad \rangle e_n = \sum_{n\in\mathbb{N}} \underbrace{\qquad}_{?} e_n.$$

Exercise 6. Let $b \in \ell^2$ and $k \in \mathbb{N}$ such that

$$\forall j \in \mathbb{N} \qquad \langle e_j, b\rangle = 0.$$

Find a formula for b.

Definition and basic properties of the multiplication operator (review)

Exercise 7 (point-wise product of sequences)**.** Let $x, y \in \mathbb{C}^{\mathbb{N}}$. The *point-wise product* or *component-wise product* of the sequences x and y is defined by:

$$\forall n \in \mathbb{N} \qquad (xy)_n := \underbrace{\hspace{2.5em}}_{?}, \qquad \text{or} \qquad xy := \big(\underbrace{\hspace{2.5em}}_{?}\big)_{n\in\mathbb{N}}.$$

Definition 1 (definition of the multiplication operator)**.** Let $a \in \ell^\infty$. The *multiplication operator* by the sequence a is an operator $\ell^2 \to \ell^2$ defined by:

$$\forall x \in \ell^2 \qquad M_a x := \underbrace{\hspace{3em}}_{?}$$

that is,

$$\forall x \in \ell^2 \qquad \forall j \in \mathbb{N} \qquad (M_a x)(j) := \underbrace{\hspace{3em}}_{?}.$$

Exercise 8 (norm of the multiplication operator)**.** Let $a \in \ell^\infty$. Then M_a is bounded and

$$\|M_a\| = \underbrace{\hspace{3em}}_{?}$$

Exercise 9 (algebraic operations with multiplication operators, review)**.** Let $a, b \in \ell^\infty$ and $\lambda \in \mathbb{C}$. Recall the formulas:

$$M_a + M_b = \qquad\qquad\qquad \lambda M_a =$$

$$\big(M_a\big)^* = \qquad\qquad\qquad M_a M_b =$$

Invertibility and spectrum of the multiplication operator (review)

Definition 2. Let $a \in \ell^\infty$. Denote by $\mathcal{CR}(a)$ the closure of the range of a.

Exercise 10 (criterion of invertibility of the multiplication operator)**.** Let $a \in \ell^\infty$. Then the following three conditions are equivalent:

$$M_a \text{ is invertible} \qquad \Longleftrightarrow \qquad \inf_{n\in\mathbb{N}} |a_n| \underbrace{\qquad\quad}_{?} \qquad \Longleftrightarrow \qquad 0 \notin \underbrace{\qquad\quad}_{?}.$$

If these conditions are fulfilled, then

$$M_a^{-1} = \underbrace{\qquad\quad}_{?}.$$

Exercise 11 (spectrum of the multiplication operator)**.** Let $a \in \ell^\infty$. Then

$$\operatorname{Sp}(M_a) = \underbrace{\qquad\quad}_{?}.$$

Multiplication operator and canonical base

Exercise 12. Let $a \in \ell^\infty$, $x \in \ell^2$ and $n \in \mathbb{N}$. Calculate:

$$\langle e_n, M_a x\rangle =$$

Exercise 13. Let $a \in \ell^\infty$ and $x \in \ell^2$. Prove that

$$M_a x = \sum_{j\in\mathbb{N}} a_j \langle e_j, x\rangle\, e_j.$$

Exercise 14. Let $a \in \ell^\infty$ and $j \in \mathbb{N}$. Calculate:

$$\langle e_j, M_a e_j\rangle =$$

Exercise 15. Let $a \in \ell^\infty$ and the sequence $(\lambda_j)_{j\in\mathbb{N}}$ is defined by:

$$\forall j \in \mathbb{N} \qquad \lambda_j := \langle e_j, M_a e_j\rangle.$$

Express a through λ_j.

Exercise 16. Let $a \in \ell^\infty$ and $j, k \in \mathbb{N}$ such that $j \neq k$. Calculate:

$$\langle e_j, M_a e_k\rangle =$$

Criteria of multiplication operator on ℓ^2

Exercise 17. Let $k \in \mathbb{N}$ and $A\colon \ell^2 \to \ell^2$ be a bounded linear operator such that

$$\forall j \in \mathbb{N} \setminus \{k\} \qquad \langle e_j, Ae_k \rangle \;=\; 0.$$

Find a formula for Ae_k.

Exercise 18. Let $A\colon \ell^2 \to \ell^2$ be a bounded linear operator. Prove that the following conditions are equivalent:

(a) There exists $a \in \ell^\infty$ such that $A = M_a$.

(b) There exists $a \in \ell^\infty$ such that for all $x \in \ell^2$,

$$Ax = \sum_{j\in\mathbb{N}} a_j \langle e_j, x\rangle\, e_j.$$

(c) For all $j, k \in \mathbb{N}$ with $j \neq k$,

$$\langle e_j, Ae_k \rangle \;=\; 0.$$