Associated Laguerre polynomials: from the Rodrigues representation to the explicit formula

Objectives. Calculate the coefficients of the associated Laguerre polynomials $L_n^{(m)}$ starting from the Rodrigues representation:

$$L_n^{(m)}(x) \coloneqq \frac{1}{n!}\, x^{-m}\, \mathrm{e}^{x}\ \frac{d^n}{dx^n}\Big(\mathrm{e}^{-x}\ x^{n+m}\Big).$$

Requirements. General Leibniz rule, factorials.

Some products and factorials (short review)

Exercise 1. Simplify:

$$\frac{7!}{4!} = \qquad\qquad \frac{(n+2)!}{n!} =$$

Exercise 2. Express the following product as a ratio of two factorials:

$$\prod_{k=4}^{11} k = 4 \cdot 5 \cdots 11 =$$

Exercise 3. Let $p, q \in \{1, 2, \ldots\}$, $p < q$. Express the following product as a ratio of two factorials:

$$\prod_{k=p}^{q} k = p \cdots q =$$

Derivatives of the exponential and monomial functions

Exercise 4. Calculate the first three derivatives of x^m ($m \geq 2$):

$$\frac{d}{dx}\big(x^m\big) = \big(x^m\big)' =$$

$$\frac{d^2}{dx^2}\big(x^m\big) = \big(x^m\big)'' =$$

$$\frac{d^3}{dx^3}\big(x^m\big) = \big(x^m\big)''' =$$

In the last formula write the coefficient as a ratio of two factorials:

$$\frac{d^3}{dx^3}\big(x^m\big) =$$

Exercise 5. Calculate the kst derivative of x^m ($k \leq m$). Write the coefficient as a ratio of two factorials.

$$\frac{d^k}{dx^k}\big(x^m\big) = \big(x^m\big)^{(k)} =$$

Exercise 6. Calculate the kst derivative of e^{ax}, where a is a parameter:

$$\frac{d^k}{dx^k}\big(\mathrm{e}^{ax}\big) =$$

Derivatives of the product of the exponential function by the monomial function

Exercise 7. Calculate the first three derivatives of the product fg of two sufficiently smooth functions:

$$(fg)' =$$

$$(fg)'' =$$

$$(fg)''' =$$

Exercise 8. Write the general Leibniz rule:

$$(fg)^{(n)} = \sum_{k=}$$

Exercise 9. Expand the following derivative. Write the sum in such an order that the powers of x form an increasing sequence. Then factorize the exponential function and the maximal possible power of the monomial:

$$\Big(\mathrm{e}^{ax}\ x^p\Big)' = \underbrace{\qquad}_{?}\ \mathrm{e}^{ax}\ x^{p-1} + \underbrace{\qquad}_{?}\ \mathrm{e}^{ax}\ x^p = \mathrm{e}^{ax}\ \underbrace{\qquad}_{?}\Big(\qquad\qquad\qquad\qquad\Big).$$

Exercise 10. Expand the following derivative using the general Leibniz rule. Write the sum in such an order that the powers of x form an increasing sequence. Then factorize the exponential function and the maximal possible power of the monomial:

$$\frac{d^n}{dx^n}\Big(\mathrm{e}^{ax}\ x^p\Big) = \sum_{k=}$$

$$= \mathrm{e}^{ax}\ \underbrace{\qquad}_{x^?}\ \sum_{k=}\ \underline{\qquad\qquad\qquad}\ x^k.$$

Definition of the associated Laguerre polynomials

Definition 1 (associated Laguerre polynomials)**.** For all $n, m \in \{0, 1, 2, \ldots\}$, define the *associated Laguerre polynomial* $L_n^{(m)}$ by the formula:

$$L_n^{(m)}(x) := \frac{1}{n!}\, x^{-m}\, \mathrm{e}^{x}\, \frac{d^n}{dx^n}\Big(\mathrm{e}^{-x}\; x^{n+m}\Big).$$

Note 1. Formulas defining some polynomials in this manner (through nst derivatives of some products) are called *Rodrigues representation* or *Rodrigues formulas.*

Exercise 11. Expand the following derivative using the result of the Exercise 10. Factorize from the sum the exponential function and the maximal possible power of the monomial.

$$\frac{d^n}{dx^n}\Big(\mathrm{e}^{-x}\; x^{n+m}\Big) = \mathrm{e}^{-x} \qquad\qquad \sum$$

Exercise 12. Write the associated Laguerre polynomial $L_n^{(m)}$ in the explicit form $\sum_{k=0}^{?} ?\; x^k$:

$$L_n^{(\alpha)}(x) =$$

Exercise 13. Express the following derivative through the associated Laguerre polynomial ($n \le p$):

$$\frac{d^n}{dx^n}\Big(\mathrm{e}^{-x}\; x^{p}\Big) =$$

Exercise 14. Let $n < p$ and $h(x) := \dfrac{d^n}{dx^n}\Big(\mathrm{e}^{-x}\; x^{p}\Big)$. Calculate:

$$h(0) = \underbrace{\qquad}_{?} \qquad\qquad \lim_{x\to+\infty} h(x) = \underbrace{\qquad}_{?}$$