Laplace transform of $\dfrac{d^n}{dt^n}\left(\mathrm{e}^{-t}\ t^m\right)$

Objectives. Recall the definition and some basic properties of the Laplace transform. Calculate the Laplace transform of the following function ($n < m$):

$$\frac{d^n}{dt^n}\left(\mathrm{e}^{-t}\ t^m\right).$$

Requirements. Integration by parts, change of variables, gamma function, associated Laguerre polynomials (Rodrigues' representation and explicit formula).

Definition of the Laplace transform

Exercise 1. Denote $(0, +\infty)$ by $\mathbb{R}_+$. Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ be a bounded continuous function. Recall the definition of its Laplace transform $\mathcal{L}f\colon \mathbb{R}_+ \to \mathbb{C}$:

$$(\mathcal{L}f)(s) := \int f(t) \qquad\quad dt.$$

Note 1. The Laplace transform is defined not only for bounded continuous functions, but for our purposes it is sufficient to consider this special case. The Laplace transform $\mathcal{L}f$ is defined not only on $\mathbb{R}_+$, but for our purposes it is sufficient to define it on $\mathbb{R}_+$.

Note 2. Because of some applications of the Laplace transform, the domain of the original function f is called the *time domain*, and the domain of the function $\mathcal{L}f$ is called the *frequency domain*. The corresponding variables will be notated by t and s.

Exercise 2. Calculate the integral:

$$\int_0^{+\infty} \mathrm{e}^{-x}\ dx =$$

Exercise 3. Calculate the Laplace transform of the function $t \mapsto \mathrm{e}^{-t}$:

$$\int \mathrm{e}^{-t} \qquad\quad dt =$$

Frequency differentiation

Exercise 4. Let $F := \mathcal{L}f$. Express $F'(s)$ as the Laplace transform of a certain function.

$$F'(s) = \int \frac{d}{ds}\Big(\qquad\qquad\qquad\qquad \Big)\,dt = \int \qquad\qquad\qquad\qquad dt.$$

Exercise 5. Let $m \in \{1, 2, \ldots\}$. Express the Laplace transform of the function $t \mapsto t^m f(t)$ through the function $F := \mathcal{L}f$.

$$\int t^m\, f(t) \qquad\qquad dt =$$

Exercise 6. Let $m \in \{1, 2, \ldots\}$. Calculate the Laplace transform of the function $t \mapsto t^m\, \mathrm{e}^{-t}$ using the results of the previous exercises:

$$\int t^m\, \mathrm{e}^{-t} \qquad\qquad dt =$$

Exercise 7. Recall the definition of the gamma function:

$$\Gamma(x) = \int \qquad\qquad\qquad dt.$$

Exercise 8. Let $\alpha, \beta > 0$. Express the following integral through the gamma function using a suitable change of variables:

$$\int_0^{+\infty} t^\alpha\, \mathrm{e}^{-\beta t}\, dt =$$

Exercise 9. Let $m \in \{1, 2, \ldots\}$. Calculate the Laplace transform of the function $t \mapsto t^m\, \mathrm{e}^{-t}$ using the gamma function. Compare the answer with the result of the Exercise 6.

$$\int_0^{+\infty} t^m\, \mathrm{e}^{-t} \qquad\qquad dt =$$

Time differentiation

Exercise 10. Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ be a continuously differentiable function such that f and f' are bounded on $\mathbb{R}_+$. Moreover suppose that the limit

$$f(0^+) \coloneqq \lim_{t\to 0^+} f(t)$$

exists and is finite. Denote the Laplace transform of f by F. Integrating by parts calculate the Laplace transform of f'.

$$\int f'(t) \qquad\qquad dt =$$

Exercise 11. Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ be a twice continuously differentiable function such that f, f' and f'' are bounded on $\mathbb{R}_+$. Moreover suppose that the limits

$$f(0^+) \coloneqq \lim_{t\to 0^+} f(t), \qquad f'(0^+) \coloneqq \lim_{t\to 0^+} f'(t)$$

exist and are finite. Denote the Laplace transform of f by F. Calculate the Laplace transform of f''.

$$\int f''(t) \qquad\qquad dt =$$

Exercise 12. Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ be a n times continuosly differentiable function such that $f, f', \ldots, f^{(n)}$ are bounded on $\mathbb{R}_+$. Moreover suppose that the limits

$$f(0^+) \coloneqq \lim_{t\to 0^+} f(t), \qquad f'(0^+) \coloneqq \lim_{t\to 0^+} f'(t), \qquad \ldots, \qquad f^{(n-1)}(0^+) \coloneqq \lim_{t\to 0^+} f^{(n-1)}(t)$$

exist and are finite. Denote the Laplace transform of f by F. Generalazing the results of the previous exercises write a formula for the Laplace transform of $f^{(n)}$.

$$\int f^{(n)}(t) \qquad dt = \tag{1}$$

Exercise 13. Prove by mathematical induction the formula (1).

Exercise 14. Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ be a n times continuosly differentiable function such that $f, f', \ldots, f^{(n)}$ are bounded on $\mathbb{R}_+$. Moreover suppose that the following limits are zero:

$$\lim_{t\to 0^+} f(t) = 0, \qquad \lim_{t\to 0^+} f'(t) = 0, \qquad \ldots, \qquad \lim_{t\to 0^+} f^{(n-1)}(t) = 0.$$

Denote the Laplace transform of f by F. Calculate the Laplace transform of $f^{(n)}$.

$$\int f^{(n)}(t) \qquad dt =$$

Laplace transform of $\dfrac{d^n}{dt^n}\left(e^{-t}\, t^m\right)$

Let $m \in \{1, 2, \ldots\}$. In the exercises of this section we work with the function $h\colon \mathbb{R}_+ \to \mathbb{C}$ defined by

$$h(t) := t^m\, e^{-t}\,.$$

Definition 1 (associated Laguerre polynomials)**.** The *associated Laguerre polynomials* $L_n^{(p)}$ can be defined by the following formula (*Rodrigues' representation*):

$$L_n^{(p)}(t) := \frac{1}{n!}\, t^{-p}\, e^{t}\, \frac{d^n}{dt^n}\Big(e^{-t}\, t^{n+p}\Big). \tag{2}$$

Note that in the notation $L_n^{(p)}$ the superscript ${}^{(p)}$ does not refer to the p-st derivative.

Exercise 15. Recall the Leibniz rule for the second derivative of the product of two functions:

$$(fg)'' =$$

Exercise 16. Calculate $L_2^{(p)}$ (apply the Leibniz rule and simplify the result):

$$L_2^{(p)}(t) = \frac{1}{2}\, t^{-p}\, e^{t}\, \Big(e^{-t}\, t^{p+2}\Big)'' =$$

Note 3. Applying the Leibniz rule for the nth derivative of the product of two functions one can easily prove that the function $L_n^{(p)}$ defined by is a polynomial and calculate its coefficients. We shall not do it here.

Exercise 17. Generalazing the result of the exercise 16, complete the following statement:

The function $L_n^{(p)}$ defined by (2) is a polynomial of degree $\underbrace{\qquad}_{?}$.

Exercise 18. Recall the definitions of h and $L_n^{(p)}$:

$$h(t) = \qquad\qquad\qquad\qquad L_n^{(p)}(t) =$$

Exercise 19. For every $n \in \{0, 1, 2, \ldots, m-1\}$ express $h^{(n)}$ through some associated Laguerre polynomial.

$$h^{(n)}(t) =$$

Exercise 20. Let $n \in \{0, 1, 2, \ldots, m-1\}$. Calculate the limit:

$$h^{(n)}(0^+) := \lim_{t \to 0^+} h^{(n)}(t) =$$

Exercise 21. Let $n \in \{0, 1, 2, \ldots, m\}$. Calculate the limit:

$$h^{(n)}(+\infty) := \lim_{t \to +\infty} h^{(n)}(t) =$$

Exercise 22. Let $n \in \{0, 1, 2, \ldots, m\}$. Determine if $h^{(n)}$ is bounded or not.

Exercise 23. Let $n \in \{0, 1, 2, \ldots, m\}$. Calculate the Laplace transform of $h^{(n)}$:

$$\int \frac{d^n}{dt^n}\left(\mathrm{e}^{-t}\, t^m\right) \qquad dt =$$