Cauchy–Riemann equations

Objectives. Every complex function f can written as

$$f(x + \mathrm{i}\, y) = u(x, y) + \mathrm{i}\, v(x, y).$$

Assuming that f is holomorph (that is, f is complex derivable in an open subset of $\mathbb{C}$) we shall express the partial derivatives of u and v in terms of f', and vise versa.

Identification of $\mathbb{R}^2$ with $\mathbb{C}$. Each point $(x, y) \in \mathbb{R}^2$ es identified with the complex number $z = x + \mathrm{i}\, y$. Every set $D \subset \mathbb{R}^2$ is identified with the set

$$\big\{x + \mathrm{i}\, y\colon\ (x, y) \in \mathbb{R}^2\big\}.$$

1. Real representation of a complex function. Let $D \subset \mathbb{C}$ and $f\colon D \to \mathbb{C}$. Put

$$u(x, y) \coloneqq \operatorname{Re}\big(f(x + \mathrm{i}\, y)\big), \qquad v(x, y) \coloneqq \operatorname{Im}\big(f(x + \mathrm{i}\, y)\big).$$

Check that

$$f(x + \mathrm{i}\, y) = u(x, y) + \mathrm{i}\, v(x, y).$$

2. Partial derivatives of the real and imaginary parts of a holomorph function. Let D be an open set in $\mathbb{C}$ and let $g\colon H(D)$, that is, for each $z_0 \in \mathbb{C}$ there exists a limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Let u, v be the functions defined in the previous exercise. Express their partial derivatives u_x, u_y, v_x, v_y in terms of the derivative f' of the function f.

3. Cauchy–Riemann equatins. Prove that

$$u_x = v_y, \qquad u_y = -v_x.$$

4. Formula for the derivative of a holomorph function in terms of the partial derivatives of its real and imaginary parts. Prove that

$$f'(x + \mathrm{i}\, y) = u_x(x, y) + \mathrm{i}\, v_x(x, y).$$