DST-I and tridiagonal symmetric Toeplitz matrices

Objectives. Diagonalize tridiagonal symmetric Toeplitz matrices using DST-I.

Requirements. DST-I.

DST-I (review)

1. Recall the definition of the matrix DST-I.

$$\mathcal{S}_n := \sqrt{\frac{2}{n+1}} \Big[\qquad \Big]_{j,k=0}^{n-1} = \sqrt{\frac{2}{n+1}} \Big[\qquad \Big]_{j,k=1}^{n}.$$

2. For each $j \in \{0, 1, \ldots, n-1\}$ denote by $\mathcal{S}_{n,j}$ the j-st column of the matrix $\mathcal{S}_n$:

$$\mathcal{S}_{n,j} := \sqrt{\frac{2}{n+1}} \Big[\qquad \Big]_{k=0}^{n-1}.$$

3. Recall the formula for the inner product of $\mathcal{S}_{n,p}$ by $\mathcal{S}_{n,q}$:

$$\langle \mathcal{S}_{n,p}, \mathcal{S}_{n,q} \rangle =$$

4. Recall the principal properties of the matrix $\mathcal{S}_n$:

$$\mathcal{S}_n^\top = \qquad , \qquad \mathcal{S}_n^* = \qquad , \qquad \mathcal{S}_n^2 = \qquad .$$

Application of DST-I to tridiagonal symmetric Toeplitz matrices

5. Define the shift matrix G_3 as

$$G_3 := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Write the formula for the (j,k)-st entry of G_n is

$$(G_n)_{j,k} := \begin{cases} 1, & \text{if} \\ 0, & \text{if} \end{cases}$$

6. **D.**enote by A_n the $n \times n$ Toeplitz matrix with generating symbol $t + t^{-1}$, i.e. the matrix whose entries are

$$(A_n)_{j,k} := \begin{cases} 1, & \text{if } j - k = 1; \\ 1, & \text{if } j - k = -1; \\ 0, & \text{otherwise.} \end{cases}$$

Write A_4 in the explicit form.

7. Express A_n through G_n and $G_n^\top$.

8. Calculate $G_n \mathcal{S}_{n,p}$.

9. Calculate $G_n^\top \mathcal{S}_{n,p}$.

10. Calculate $A_n \mathcal{S}_{n,p}$.

11. Find the eigenvalues and the eigenvectors of A_n.

12. Calculate $\mathcal{S}_n A_n \mathcal{S}_n$.