Discrete Sine Transform I

Objectives. Study the Discrete Sine Transform (variant I) and its principal properties.

Requirements. Sums, Euler's formulas, trigonometric identities, matrices.

Geometric progression with exponential function (review)

1. Recall the formula:

$$\sum_{k=0}^{n-1} q^k = \begin{cases} \qquad\qquad , & \text{if } q \\ \quad , & \text{if } q \end{cases}$$

2. Let $\alpha \in \mathbb{R}$. Recall when $\mathrm{e}^{i\alpha} = 1$.

$$\mathrm{e}^{i\alpha} = 1 \qquad \iff$$

3. Let $\alpha \in \mathbb{R}$. Recall the formula for the following sum:

$$\sum_{k=0}^{n-1} \mathrm{e}^{\mathrm{i}\,k\alpha} =$$

4. Let $\alpha \in \mathbb{R}$. Using the previous result deduce a formula for the following sum:

$$\sum_{k=0}^{n-1} \cos(k\alpha) =$$

Some trigonometric identities

5. Euler's formulas. Express $\cos(\alpha)$ and $\sin(\alpha)$ through $\mathrm{e}^{\mathrm{i}\,\alpha}$ and $\mathrm{e}^{-\mathrm{i}\,\alpha}$:

$\cos(\alpha) =$ $\qquad\qquad\qquad\qquad$ $\sin(\alpha) =$

6. Product of sines. Prove the formula:

$$2\sin(\alpha)\sin(\beta) = \cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}.$$

7. Square of sine. Express through $\cos(2\alpha)$:

$$2\sin^2(\alpha) = \underbrace{\qquad\qquad}_{?}$$

8. Sum of sine products. Simplify the sum:

$$\sum_{k=0}^{n-1}\sin(k\alpha)\sin(k\beta) =$$

9. A special sum of sine squares. Let $j \in \mathbb{Z}$. Simplify:

$$\sum_{k=0}^{n-1}\sin^2\frac{jk}{n+1} =$$

10. A special sum of sine products. Let $p, q \in \{0, 1, \ldots, n-1\}$, $p \neq q$. Simplify:

$$\sum_{k=0}^{n-1}\sin\frac{pk}{n+1}\sin\frac{qk}{n+1} =$$

DST-I and its properties

11. Define the matrix $\mathcal{S}_n \in \mathcal{M}_n(\mathbb{C})$ by

$$\mathcal{S}_n := \sqrt{\frac{2}{n+1}} \left[\sin \frac{(j+1)(k+1)\pi}{n+1} \right]_{j,k=0}^{n-1}.$$

Note that this matrix also can be written in the form

$$\mathcal{S}_n = \sqrt{\frac{2}{n+1}} \left[\sin \frac{jk\pi}{n+1} \right]_{j,k=1}^{n},$$

but in this text we prefer to start indices with zero.

12. Write the matrix $\mathcal{S}_3$ in the explicit form (without calculate sines):

$$\mathcal{S}_3 = \sqrt{\frac{}{}} \left[\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & & \end{array} \right].$$

13. Write the matrix $\mathcal{S}_4$ in the explicit form (without calculate sines):

14. Calculate the transpose and the conjugated transpose of $\mathcal{S}_n$:

$$\mathcal{S}_n^\top = \underbrace{}_{?}, \qquad \mathcal{S}_n^* = \underbrace{}_{?}.$$

Columns of the matrix DST-I

15. For each $p \in \{0, 1, \ldots, n-1\}$ denote by $\mathcal{S}_{n,p}$ the p-st column of the matrix $\mathcal{S}_n$:

$$\mathcal{S}_{n,p} := \sqrt{\frac{2}{n+1}} \left[\qquad\qquad\qquad\qquad \right]_{j=0}^{n-1}.$$

16. Write the column $\mathcal{S}_{4,2}$ in the explicit form.

17. Let $x, y \in \mathbb{C}^n$. Recall the definition of the canonical inner product (= dot product) in $\mathbb{C}^n$:

$$\langle x, y \rangle :=$$

18. Let $p, q \in \{0, 1, \ldots, n-1\}$, $p \neq q$. Compute the inner product (dot product) of $\mathcal{S}_{n,p}$ and $\mathcal{S}_{n,q}$:

$$\langle \mathcal{S}_{n,p}, \mathcal{S}_{n,q} \rangle =$$

19. Let $p \in \{0, 1, \ldots, n-1\}$. Compute the square of the norm of the vector $\mathcal{S}_{n,p}$:

$$\langle \mathcal{S}_{n,p}, \mathcal{S}_{n,p} \rangle =$$

Main properties of the matrix DST-I

20. Calculate the square of the matrix $\mathcal{S}_n$, i.e. multiply $\mathcal{S}_n$ by $\mathcal{S}_n$.

21. Calculate the products:

$$\mathcal{S}_n^\top \mathcal{S}_n = \underbrace{\qquad\qquad}_{?}, \qquad \mathcal{S}_n^* \mathcal{S}_n = \underbrace{\qquad\qquad}_{?}.$$

22. Recall the definition of orthogonal matrix. Recall the definition of unitary matrix. Determine if $\mathcal{S}_n$ has these properties.