Operator Algebras, Toeplitz Operators and Related Topics

Radial operators in Fock polyanalytic spaces

Ana María Tellería Romero
Joint work with Egor Maximenko

ESFM-IPN

Boca del Río Veracruz, 2018

Object of study

Define the Gaussian weight on the Complex plane as

$$d\mu_G(z) = \frac{1}{\pi} e^{-|z|^2} d\mu(z),$$

then, the inner product in $\mathcal{L}^2(\mathbb{C}, d\mu_G)$ is

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^2} d\mu(z).$$

Define the n-th Fock space

$$F_n = \left\{ f \in \mathcal{C}^n(\mathbb{R}^2) \,\middle|\, \frac{\partial^n f}{\partial \overline{z}^n}(z) = 0, \;\; f \in \mathcal{L}^2(\mathbb{C}, d\mu_G) \right\}.$$

Monomials in z and $\overline{z}$

Let p, $q \in \mathbb{N}_0$, define the function $m_{p,q} : \mathbb{C} \to \mathbb{C}$, as

$$m_{p,q}(z) = z^p \overline{z}^q.$$

Given $d \in \mathbb{Z}$,

$$\mathcal{D}_d = \text{clos}(\text{gen}\{m_{p,q} | p - q = d\}).$$

$\mathcal{D}_0$

$z^0\overline{z}^0$	$z^0\overline{z}^1$	$z^0\overline{z}^2$	$z^0\overline{z}^3$	$\dots$
$z^1\overline{z}^0$	$z^1\overline{z}^1$	$z^1\overline{z}^2$	$z^1\overline{z}^3$	$\dots$
$z^2\overline{z}^0$	$z^2\overline{z}^1$	$z^2\overline{z}^2$	$z^2\overline{z}^3$	$\dots$
$z^3\overline{z}^0$	$z^3\overline{z}^1$	$z^3\overline{z}^2$	$z^3\overline{z}^3$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The linear span of $m_{p,q}$ is a dense subset of $\mathcal{L}^2(\mathbb{C}, d\mu_G)$.

Monomials in z and $\overline{z}$

Let p, $q \in \mathbb{N}_0$, define the function $m_{p,q} : \mathbb{C} \to \mathbb{C}$, as

$$m_{p,q}(z) = z^p \overline{z}^q.$$

Given $d \in \mathbb{Z}$,

$$\mathcal{D}_d = \mathrm{clos}(\mathrm{gen}\{m_{p,q} | p - q = d\}).$$

$\mathcal{D}_1$

$$\begin{array}{ccccc} z^0\overline{z}^0 & z^0\overline{z}^1 & z^0\overline{z}^2 & z^0\overline{z}^3 & \dots \\ z^1\overline{z}^0 & z^1\overline{z}^1 & z^1\overline{z}^2 & z^1\overline{z}^3 & \dots \\ z^2\overline{z}^0 & z^2\overline{z}^1 & z^2\overline{z}^2 & z^2\overline{z}^3 & \dots \\ z^3\overline{z}^0 & z^3\overline{z}^1 & z^3\overline{z}^2 & z^3\overline{z}^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

The linear span of $m_{p,q}$ is a dense subset of $\mathcal{L}^2(\mathbb{C}, d\mu_G)$.

Monomials in z and $\overline{z}$

Let p, $q \in \mathbb{N}_0$, define the function $m_{p,q} : \mathbb{C} \to \mathbb{C}$, as

$$m_{p,q}(z) = z^p \overline{z}^q.$$

Given $d \in \mathbb{Z}$,

$$\mathcal{D}_d = \text{clos}(\text{gen}\{m_{p,q} | p - q = d\}).$$

$\mathcal{D}_{-1}$

$$\begin{array}{ccccc} z^0\overline{z}^0 & \boxed{z^0\overline{z}^1} & z^0\overline{z}^2 & z^0\overline{z}^3 & \dots \\ z^1\overline{z}^0 & z^1\overline{z}^1 & \boxed{z^1\overline{z}^2} & z^1\overline{z}^3 & \dots \\ z^2\overline{z}^0 & z^2\overline{z}^1 & z^2\overline{z}^2 & \boxed{z^2\overline{z}^3} & \dots \\ z^3\overline{z}^0 & z^3\overline{z}^1 & z^3\overline{z}^2 & z^3\overline{z}^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

The linear span of $m_{p,q}$ is a dense subset of $\mathcal{L}^2(\mathbb{C}, d\mu_G)$.

Canonical basis of polynomials in z and $\overline{z}$

Let p, q, j, $k \in \mathbb{N}_0$, then

$$\langle m_{p,q}, m_{j,k} \rangle = \delta_{p-q,j-k}\,(p+k)!.$$

$$\mathcal{D}_d \perp \mathcal{D}_c, \qquad d \neq c.$$

Gram-Schmidt by diagonal:

$$\begin{array}{llll}
b_{0,0} = m_{0,0} & b_{0,1} = m_{0,1} & b_{0,2} = \frac{1}{\sqrt{2}} m_{0,2} & \dots \\
b_{1,0} = m_{1,0} & b_{1,1} = m_{1,1} - m_{0,0} & b_{1,2} = \frac{1}{\sqrt{2}}(m_{1,2} - 2m_{0,1}) & \dots \\
b_{2,0} = \frac{1}{\sqrt{2}} m_{2,0} & b_{2,1} = \frac{1}{\sqrt{2}}(m_{2,1} - 2m_{1,0}) & b_{2,2} = \frac{1}{2}(m_{2,2} - 4m_{1,1} + 2m_{0,0}) & \dots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

Canonical basis of polynomials in z and $\overline{z}$

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!} \frac{\partial^n}{\partial x^n}(e^{-x} x^{n+\alpha}).$$

The basis elements have explicit form

$$b_{p,q}(z) = \begin{cases} (-1)^q \sqrt{\frac{p!}{q!}}\, z^{p-q}\, L_q^{p-q}(|z|^2), & p \geqslant q, \\ (-1)^p \sqrt{\frac{q!}{p!}}\, \overline{z}^{q-p}\, L_p^{q-p}(|z|^2), & q > p. \end{cases}$$

Ali, Bagarello, Gazeau (2015),
D-Pseudo-Bosons, Complex Hermite Polynomials, and Integral Quantization

Creation and annihilation operators

$$A^\dagger = \overline{z} - \frac{\partial}{\partial z}, \qquad A = \frac{\partial}{\partial \overline{z}}.$$

Proposition.

$$A^\dagger b_{p,q} = \sqrt{q+1} b_{p,q+1}, \qquad A b_{p,q+1} = \sqrt{q+1} b_{p,q}.$$

Vasilevski (2000),
Poly-Fock Spaces.

Creation and annihilation operators

$$A^\dagger b_{2,1} = \sqrt{2!}\, b_{2,2}$$

$$\begin{array}{ccccc}
b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & \cdots \\
b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & \cdots \\
b_{2,0} & b_{2,1} \xrightarrow{\sqrt{2}} & b_{2,2} & b_{2,3} & \cdots \\
b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Creation and annihilation operators

$$Ab_{3,3} = \sqrt{3!}\,b_{3,2}$$

$$\begin{array}{ccccc}
b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & \cdots \\
b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & \cdots \\
b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & \cdots \\
b_{3,0} & b_{3,1} & b_{3,2} \xleftarrow{\sqrt{6}} & b_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Polyanalytic Fock spaces F_n

Let $n \in \mathbb{N}_0$, then

$$F_n = \left\{ f \in \mathcal{C}^n(\mathbb{R}^2) \,\middle|\, \frac{\partial^n f}{\partial \overline{z}^n}(z) = 0, \quad f \in \mathcal{L}^2(\mathbb{C}, d\mu_G) \right\}.$$

$$F_n = \operatorname{clos}(\operatorname{gen}\{b_{p,q} \mid p \in \mathbb{N}_0,\ 0 \leqslant q < n\}).$$

F_2

$m_{0,0}$	$m_{0,1}$	$m_{0,2}$	$m_{0,3}$	$\cdots$
$m_{1,0}$	$m_{1,1}$	$m_{1,2}$	$m_{1,3}$	$\cdots$
$m_{2,0}$	$m_{2,1}$	$m_{2,2}$	$m_{2,3}$	$\cdots$
$m_{3,0}$	$m_{3,1}$	$m_{3,2}$	$m_{3,3}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$b_{0,0}$	$b_{0,1}$	$b_{0,2}$	$b_{0,3}$	$\cdots$
$b_{1,0}$	$b_{1,1}$	$b_{1,2}$	$b_{1,3}$	$\cdots$
$b_{2,0}$	$b_{2,1}$	$b_{2,2}$	$b_{2,3}$	$\cdots$
$b_{3,0}$	$b_{3,1}$	$b_{3,2}$	$b_{3,3}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

True-polyanalytic Fock spaces $F_{(n)}$

$$F_{(n)} = F_n \bigcap F_{n-1}^{\perp}.$$

$$F_{(n)} = \text{clos}(\text{gen}\{b_{p,n-1} \mid p \in \mathbb{N}_0\}).$$

$$F_{(2)}$$

$$\begin{matrix} m_{0,0} & m_{0,1} & m_{0,2} & m_{0,3} & \cdots \\ m_{1,0} & m_{1,1} & m_{1,2} & m_{1,3} & \cdots \\ m_{2,0} & m_{2,1} & m_{2,2} & m_{2,3} & \cdots \\ m_{3,0} & m_{3,1} & m_{3,2} & m_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix} \qquad \begin{matrix} b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & \cdots \\ b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & \cdots \\ b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & \cdots \\ b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Bounded creation and annihilation operators

$$A_n^\dagger : F_{(n)} \to F_{(n+1)} \qquad A_n : F_{(n+1)} \to F_{(n)}$$

$$A_n^\dagger = \frac{1}{\sqrt{n+1}}\left(\overline{z} - \frac{\partial}{\partial z}\right) \qquad A_n = \frac{1}{\sqrt{n+1}}\left(\frac{\partial}{\partial \overline{z}}\right)$$

$$A_n^\dagger b_{n,p} = b_{n+1,p} \qquad A_n b_{n+1,p} = b_{n,p}$$

$$A_1^\dagger F_{(3)} = F_{(4)}$$

$$\begin{array}{cccccc}
b_{0,0} & b_{0,1} & b_{0,2} & \to & b_{0,3} & \cdots \\
b_{1,0} & b_{1,1} & b_{1,2} & \to & b_{1,3} & \cdots \\
b_{2,0} & b_{2,1} & b_{2,2} & \to & b_{2,3} & \cdots \\
b_{3,0} & b_{3,1} & b_{3,2} & \to & b_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \longrightarrow & \vdots & \ddots
\end{array}$$

Bounded creation and annihilation operators

$$A_n^\dagger : F_{(n)} \to F_{(n+1)} \qquad A_n : F_{(n+1)} \to F_{(n)}$$

$$A_n^\dagger = \frac{1}{\sqrt{n+1}}\left(\overline{z} - \frac{\partial}{\partial z}\right) \qquad A_n = \frac{1}{\sqrt{n+1}}\left(\frac{\partial}{\partial \overline{z}}\right)$$

$$A_n^\dagger b_{n,p} = b_{n+1,p} \qquad A_n b_{n+1,p} = b_{n,p}$$

$$A_2 F_{(3)} = F_{(2)}$$

$$\begin{array}{ccccc}
b_{0,0} & b_{0,1} \leftarrow b_{0,2} & & b_{0,3} & \cdots \\
b_{1,0} & b_{1,1} \leftarrow b_{1,2} & & b_{1,3} & \cdots \\
b_{2,0} & b_{2,1} \leftarrow b_{2,2} & & b_{2,3} & \cdots \\
b_{3,0} & b_{3,1} \leftarrow b_{3,2} & & b_{3,3} & \cdots \\
\vdots & \vdots \longleftarrow \vdots & & \vdots & \ddots
\end{array}$$

Reproducing kernel

Proposition.

The reproducing kernel of $F_{(n)}$ is

$$K_z^{(n)}(w) = e^{\overline{z}w} L_{n-1}(|z-w|^2).$$

Idea of proof.

$$b_{p,q-1} = A_{q-1}^{\dagger} \dots A_1^{\dagger} b_{p,0},$$

$$K_z^{(n)}(w) = \frac{1}{n-1} \left(z - \frac{\partial}{\partial \overline{z}} \right) \left(\overline{w} - \frac{\partial}{\partial w} \right) K_z^{(n-1)}.$$

Reproducing kernel

Proposition.

The reproducing kernel of the polyanalytic Fock space F_n is

$$K_z^n(w) = e^{\overline{z}w} L_{n-1}^1(|z-w|^2).$$

Idea of proof.

$$F_n = \bigoplus_{j=1}^{n} F_{(j)}, \qquad \text{and} \qquad L_n^{\alpha+1}(x) = \sum_{j=0}^{n} L_j^{\alpha}(x),$$

Rotation operators

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \quad \alpha \in \mathbb{T}, \quad R_\alpha : \mathcal{L}^2(\mathbb{C}, d\mu_G) \to \mathcal{L}^2(\mathbb{C}, d\mu_G)$$

$$(R_\alpha f)(z) = f(e^{-i\alpha} z).$$

$(R_\alpha)_{\alpha\in\mathbb{T}}$ is a unitary representation of $\mathbb{T}$ in $\mathcal{L}^2(\mathbb{C}, d\mu_G)$.

D_d is an eigensubspace under rotation operators

Lemma.
Let $\alpha \in \mathbb{R}$, $d \in \mathbb{Z}$

$$R_\alpha(\mathcal{D}_d) = \mathcal{D}_d.$$

$$R_\alpha m_{d+q,q} = e^{-id\alpha} m_{d+q,q}.$$

$m_{0,0}$	$m_{0,1}$	$m_{0,2}$	$m_{0,3}$	$\ddots$
$m_{1,0}$	$m_{1,1}$	$m_{1,2}$	$m_{1,3}$	$\ddots$
$m_{2,0}$	$m_{2,1}$	$m_{2,2}$	$m_{2,3}$	$\ddots$
$m_{3,0}$	$m_{3,1}$	$m_{3,2}$	$m_{3,3}$	$\ddots$
$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

$$R_\alpha(F_n) = F_n, \quad R_\alpha(F_{(n)}) = F_{(n)}$$

W^* algebras of radial operators

$$\mathcal{R} = \{S \in \mathcal{B}(\mathcal{L}^2(\mathbb{C}, d\mu_g)) \mid \forall\, \alpha \in \mathbb{T}, \quad R_\alpha S = S R_\alpha\},$$

W^* algebras of radial operators

$$\mathcal{R} = \{S \in \mathcal{B}(\mathcal{L}^2(\mathbb{C}, d\mu_g)) \mid \forall\, \alpha \in \mathbb{T}, \quad R_\alpha S = SR_\alpha\},$$

$$\mathcal{R}_n = \{S \in \mathcal{B}(F_n) \mid \forall\, \alpha \in \mathbb{T}, \quad R_{\alpha,n} S = SR_{\alpha,n}\},$$

$$\mathcal{R}_{(n)} = \{S \in \mathcal{B}(F_{(n)}) \mid \forall\, \alpha \in \mathbb{T}, \quad R_{\alpha,(n)} S = SR_{\alpha,(n)}\},$$

where $R_{\alpha,n} : F_n \to F_n$, and $R_{\alpha,(n)} : F_{(n)} \to F_{(n)}$.

D_d is invariant under radial operators

Proposition.
Let $S \in \mathcal{R}$, then $S(\mathcal{D}_d) \subset \mathcal{D}_d$ for all $d \in \mathbb{Z}$.

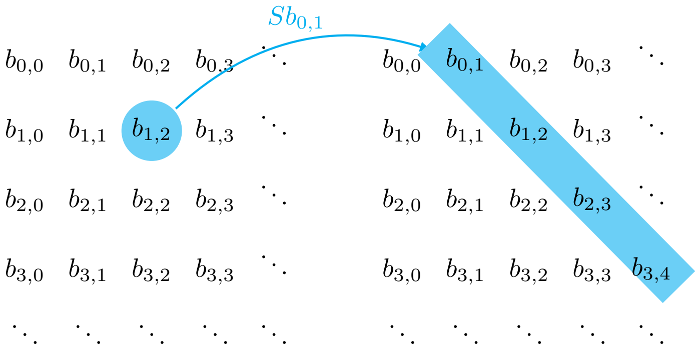
$$\mathcal{L}^2(\mathbb{C}, d\mu_g) = \bigoplus_{d \in \mathbb{Z}} \mathcal{D}_d$$

Let $c \neq d$, $f \in \mathcal{D}_d$ and $g \in \mathcal{D}_c$,

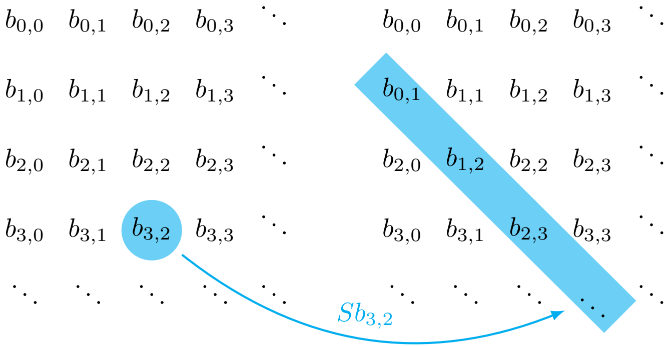
$$\langle Sf, g \rangle = \langle R_\alpha Sf, R_\alpha g \rangle = \langle S\, R_\alpha f, e^{-ic\alpha} g \rangle = e^{-i(d-c)\alpha} \langle Sf, g \rangle$$

$$(e^{-i(d-c)\alpha} - 1)\langle Sf, g \rangle = 0$$

D_d is invariant under radial operators



D_d is invariant under radial operators



Decomposition into diagonal subspaces

Define $S_d \in \mathcal{B}(\mathcal{D}_d)$, such that

$$S = \bigoplus_{d \in \mathbb{Z}} S_d$$

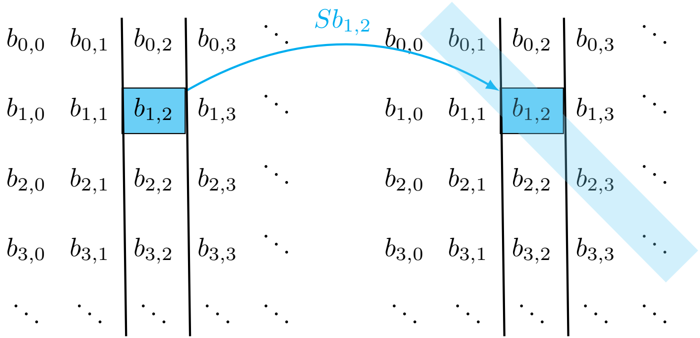
Proposition.

$$\mathcal{R} \simeq \bigoplus_{d \in \mathbb{Z}} \mathcal{B}(\mathcal{D}_d)$$

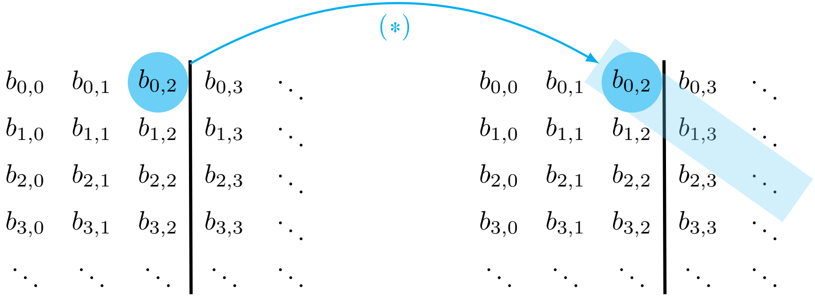
Radial operators in true-polyanalytic Fock spaces

Proposition.

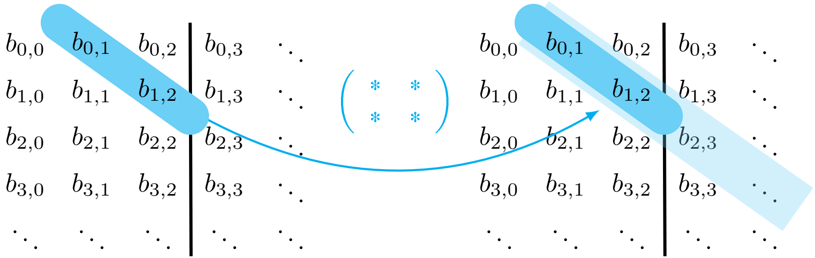
The class of radial operators in a given true Fock space $F_{(n)}$ is diagonal with respect to the basis $(b_{p,n-1})_{p\in\mathbb{N}_0}$.



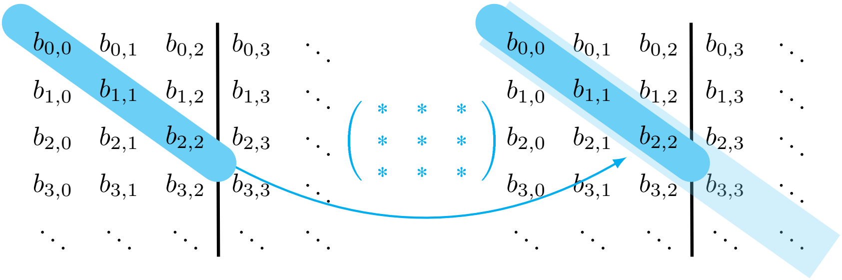
Radial operators in polyanalytic Fock spaces



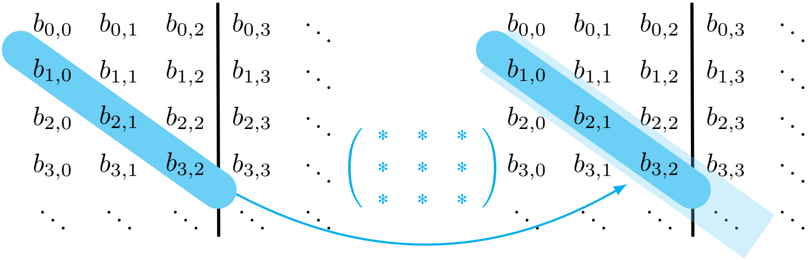
Radial operators in polyanalytic Fock spaces



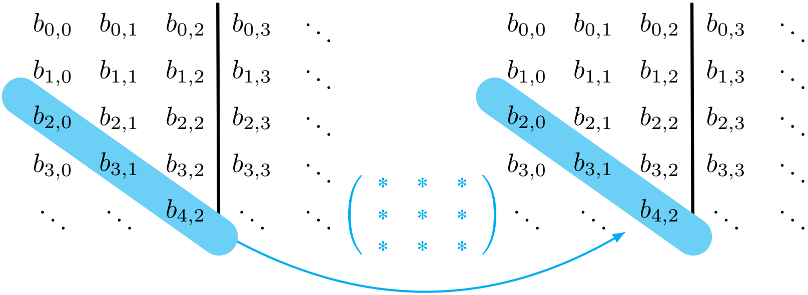
Radial operators in polyanalytic Fock spaces



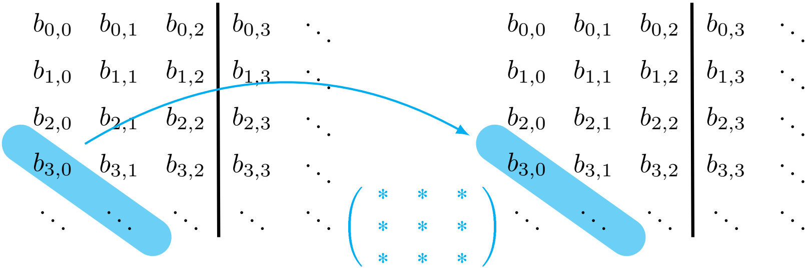
Radial operators in polyanalytic Fock spaces



Radial operators in polyanalytic Fock spaces



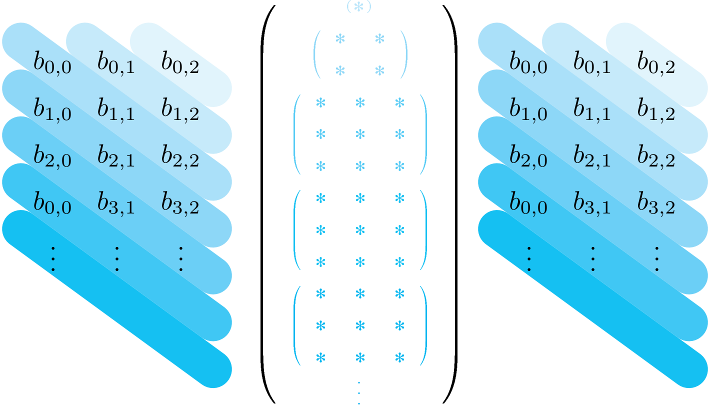
Radial operators in polyanalytic Fock spaces



Radial operators in polyanalytic Fock spaces

$$
\begin{array}{ccc|cc}
b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & \ddots \\
b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & \ddots \\
b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & \ddots \\
b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots
\end{array}
\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}
\begin{array}{ccc|cc}
b_{0,0} & b_{0,1} & b_{0,2} & b_{0,3} & \ddots \\
b_{1,0} & b_{1,1} & b_{1,2} & b_{1,3} & \ddots \\
b_{2,0} & b_{2,1} & b_{2,2} & b_{2,3} & \ddots \\
b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots
\end{array}
$$

Radial operators in polyanalytic Fock spaces



Radial operators in polyanalytic Fock spaces

$$\mathcal{R}_n \simeq \mathbb{M}_n = \bigoplus_{d=-n+1}^{\infty} \mathcal{M}_{\min\{n,n+d\}}$$

$$\begin{pmatrix} (*) \\ \begin{pmatrix} * & * \\ * & * \end{pmatrix} \\ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \\ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \\ \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \\ \vdots \end{pmatrix}$$

Future work

Toeplitz operators with radial symbols

$F_{(n)}$ Esmeral and Maximenko, 2015 $RT_{(1)} \simeq$ square root oscillating sequences.
CONJECTURE: $RT_{(n)} \simeq$ square root oscillating sequences.

F_n CONJECTURE: Toeplitz operators with radial symbol that have limit $\simeq$ matrix sequences with scalar limits.

Thank you!