La transformada de ondícula continua y algunas clases de operadores de localización

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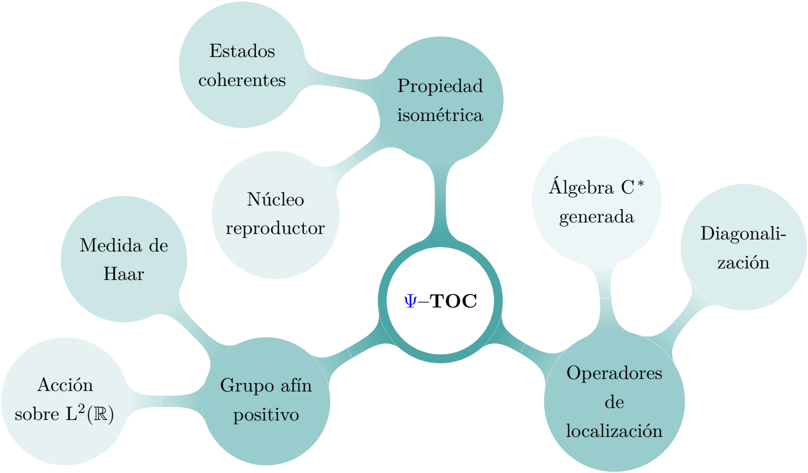
Contenido

- El grupo afín positivo
- La transformada de ondícula continua y su propiedad isométrica
- Los operadores de localización

Panorama general de la presentación



Traslaciones, dilataciones y transformaciones afines

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Transformación afín positiva $A_{\lambda,a}$

$(\lambda \in \mathbb{R}^+,\ a \in \mathbb{R})$

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \mapsto & \lambda\, x + a \end{array}$$

Composición de dos transformaciones afines:

$$\boxed{A_{\lambda,a} \circ A_{\mu,b} = A_{\lambda\mu,\lambda b+a}}$$

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Observación:

$$\begin{aligned} A_{1,0} &= \mathrm{I}_{\mathbb{R}} \\ A^{-1}_{\lambda,a} &= A_{\frac{1}{\lambda},-\frac{a}{\lambda}} \end{aligned}$$

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$(\mathbb{G}, \cdot\,)$ es un grupo

$(\mathbb{G} = \mathbb{R}^+ \rtimes \mathbb{R})$

Acción del grupo $\mathbb{G}$ sobre $L^2(\mathbb{R})$

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$$\left.\begin{array}{r} \psi \in L^2(\mathbb{R}) \\ (\lambda, a) \in \mathbb{G} \end{array}\right\} \longrightarrow \psi_{\lambda,a}(x)$$

Acción del grupo $\mathbb{G}$ sobre $\mathrm{L}^2(\mathbb{R})$

$$\left.\begin{array}{r} \psi \in \mathrm{L}^2(\mathbb{R}) \\ (\lambda, a) \in \mathbb{G} \end{array}\right\} \longrightarrow \psi_{\lambda,a}(x) = \frac{1}{\sqrt{\lambda}}\, \psi\left(\frac{x-a}{\lambda}\right) \quad \forall\, x \in \mathbb{R}.$$

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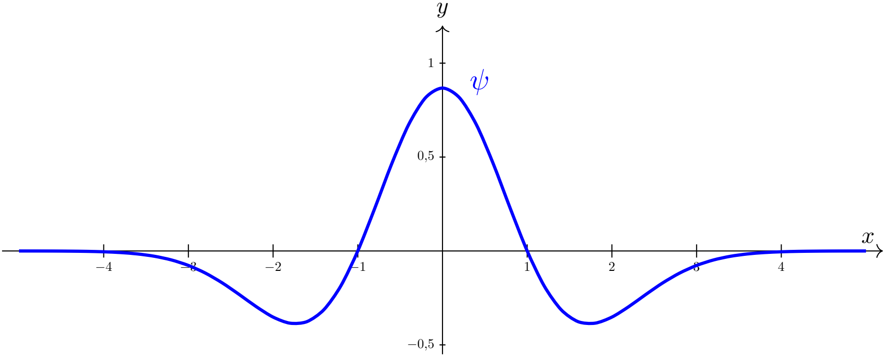
$$\psi_{\frac{1}{\lambda},0} = \sqrt{\lambda}\,\psi(\lambda x)$$

Ejemplo: Para cierta función ψ tenemos



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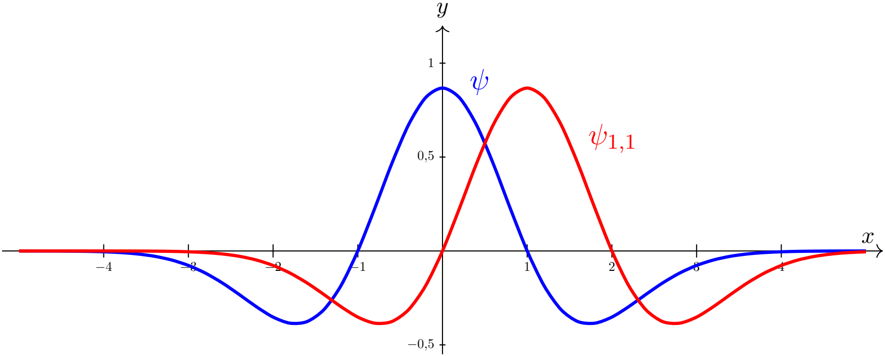

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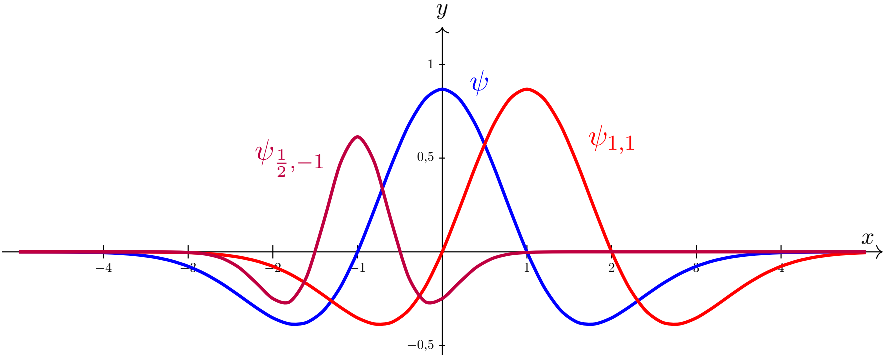

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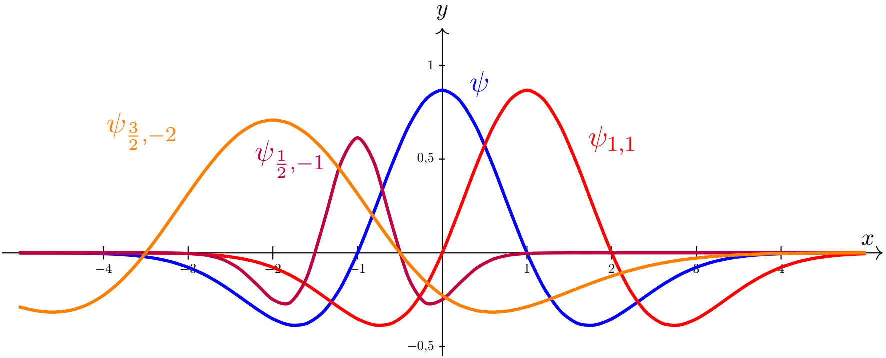
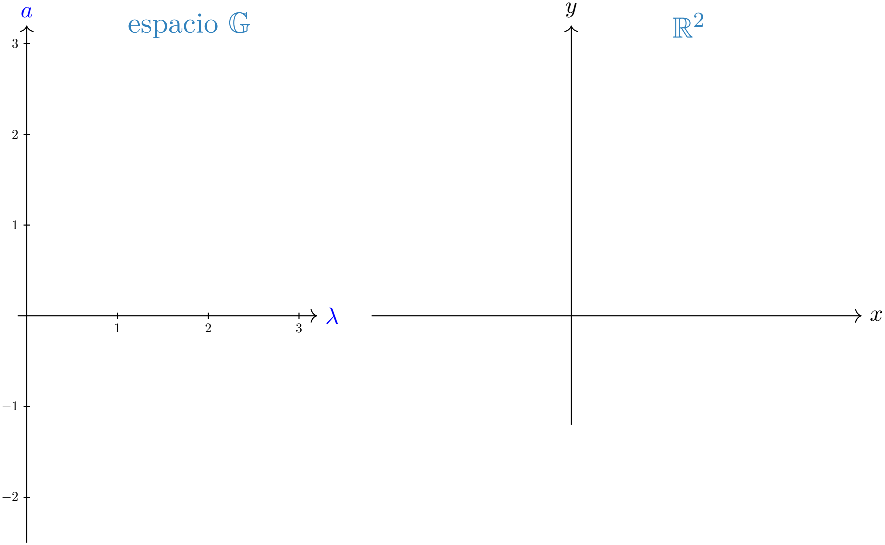
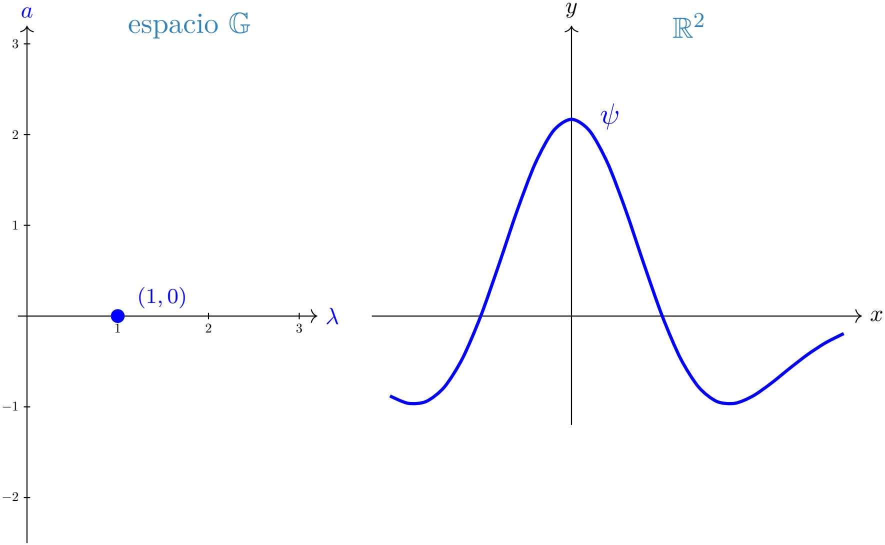




espacio $\mathbb{G}$
a
λ
1
2
3
−1
−2
$\mathbb{R}^2$
x
y



a
espacio $\mathbb{G}$
3
2
1
$(1,0)$
λ
1
2
3
−1
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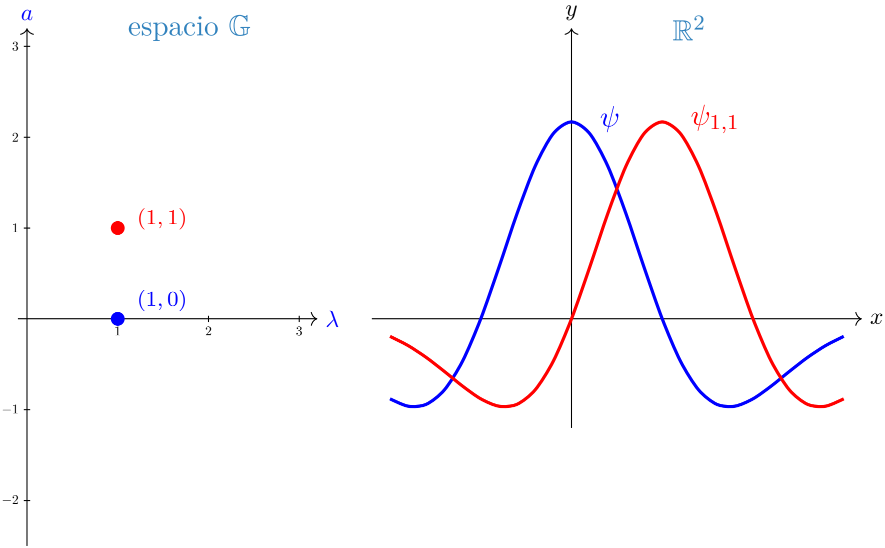
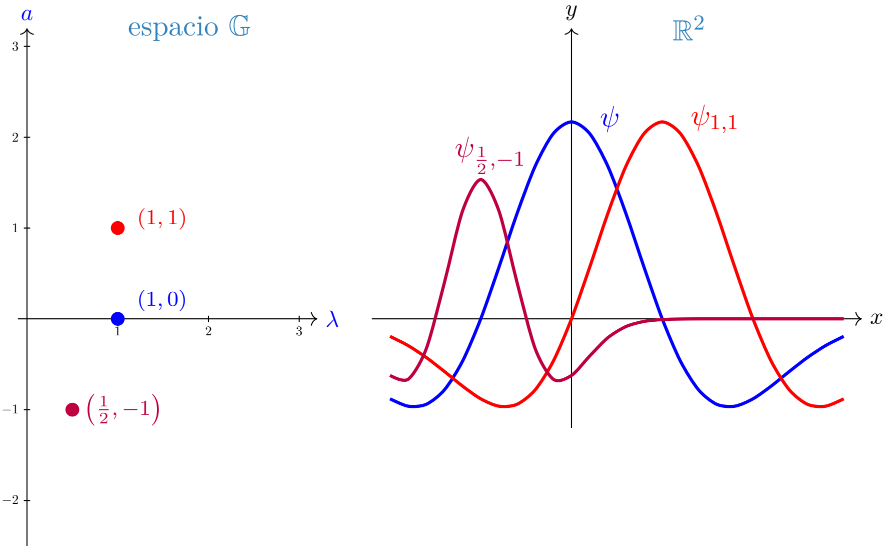
y
$\mathbb{R}^2$
ψ
x



a
espacio 𝔾
3
2
1
(1, 1)
(1, 0)
−1
−2
1
2
3
λ
y
ℝ²
ψ
ψ1,1
x



a
espacio $\mathbb{G}$
3
2
1
$(1,1)$
$(1,0)$
λ
1
2
3
-1
$\left(\frac{1}{2},-1\right)$
-2
y
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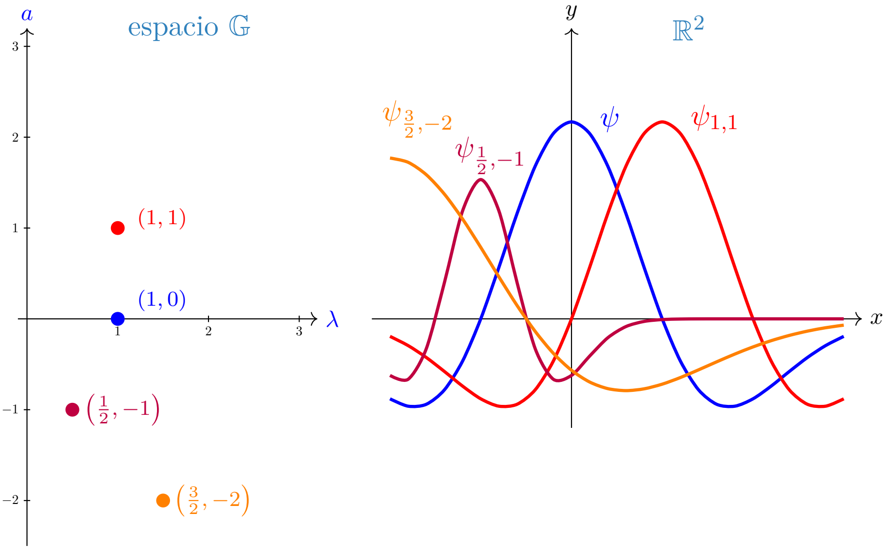
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Una medida de Haar izquierda en $\mathbb{G}$

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ν_L es una medida *invariante* bajo traslaciones por la izquierda:

$$\nu_L((\lambda, a)X) = \nu_L(X)$$

Ondícula (def)

Es una función $\Psi \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R})$ tal que

(Condición de admisibilidad)

$$\int_{\mathbb{R}^+} \frac{|\widehat{\Psi}(\lambda a)|^2}{\lambda}\, \mathrm{d}\lambda = 1, \quad \forall\, a \in \mathbb{R} \setminus \{0\}$$

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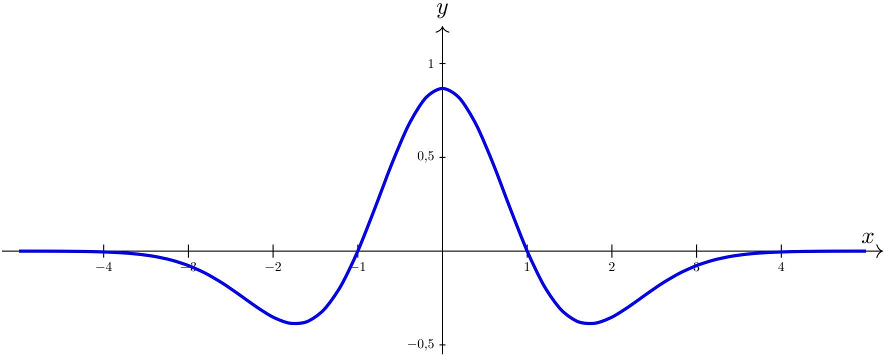
o equivalentemente

$$\int_{\mathbb{R}^+} \frac{|\widehat{\Psi}(-\lambda)|^2}{\lambda}\, \mathrm{d}\lambda = \int_{\mathbb{R}^+} \frac{|\widehat{\Psi}(\lambda)|^2}{\lambda}\, \mathrm{d}\lambda = 1$$

Ejemplo: ondícula *sombrero mexicano*

$$\Psi(x) = \frac{2}{\sqrt{3}\,\pi^{1/4}}\left(1 - x^2\right) e^{-\frac{1}{2}x^2}.$$



La transformada de ondícula continua

relativa a la ondícula Ψ

(Ψ–TOC)

$$\begin{aligned} \mathcal{W}_\Psi : \mathrm{L}^2(\mathbb{R}) &\longrightarrow L^2(\mathbb{G}, \nu_L) \\ \mathcal{W}_\Psi f(\lambda, a) &= \langle\, f \,,\, \Psi_{\lambda,a} \,\rangle_{\mathrm{L}^2(\mathbb{R})} \end{aligned}$$

Ejemplo

- ☛ $f(x) = \operatorname{sen}(x)\, e^{-2x^2}$
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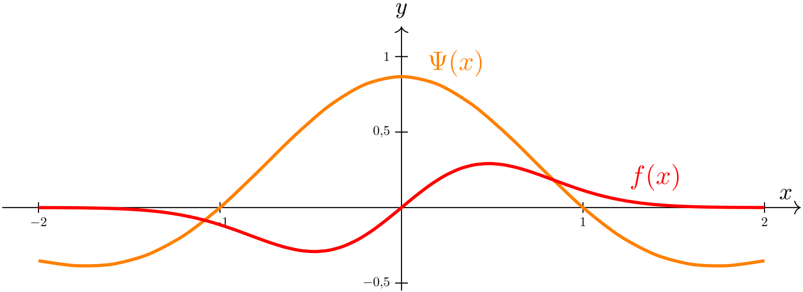
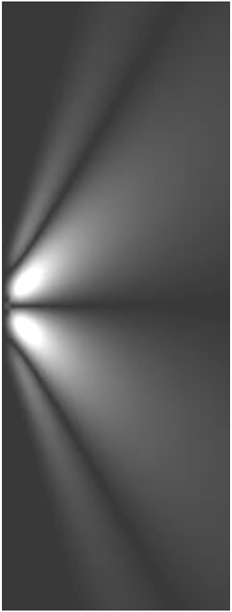
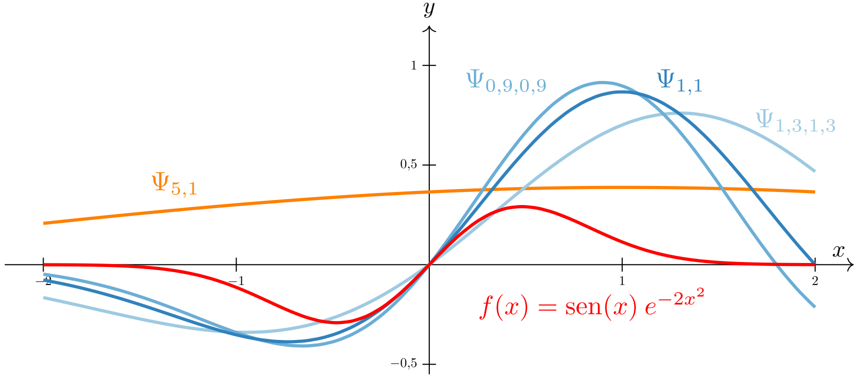


Resultado de aplicar Ψ–TOC a la función f



Comparación entre algunas ondículas hijas y la función f



Ψ–TOC en términos de la transformada de Fourier

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$$1) \quad \mathcal{F}[\Psi_{\lambda,a}](\xi) \quad = \quad \int_{\mathbb{R}} \Psi_{\lambda,a}(x)\, e^{-2\pi i x \xi}\, \mathrm{d}x$$

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$$\Big/\ \begin{matrix} x = \lambda y + a \\ \mathrm{d}x = \lambda\, \mathrm{d}y \end{matrix}\ \Big/ \quad = \sqrt{\lambda} \int_{\mathbb{R}} \Psi(y)\, e^{-2\pi i(\lambda y + a)\xi}\, \mathrm{d}y$$

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&= \sqrt{\lambda} \int_{\mathbb{R}} \Psi(y)\, e^{-2\pi i(\lambda y + a)\xi}\, \mathrm{d}y \\
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$$\boxed{\mathcal{F}[\Psi_{\lambda,a}](\xi) = \widehat{\Psi}_{\frac{1}{\lambda},0}(\xi)\, e^{-2\pi i a\xi}}$$

Ψ–TOC en términos de la transformada de Fourier

2) $$\mathcal{W}_{\Psi} f(\lambda, a) = \langle\, f \,,\, \Psi_{\lambda,a} \,\rangle \;=\; \langle\, \widehat{f}\,,\, \mathcal{F}[\Psi_{\lambda,a}]\,\rangle$$

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$$\| N_\Psi g \|^2_{L^2(\mathbb{G})} = \int_{\mathbb{R}} \int_{\mathbb{R}^+} \left| g(a)\, \overline{\widehat{\Psi}_{\frac{1}{\lambda},0}(a)} \right|^2 \frac{d\lambda\, da}{\lambda^2}$$

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$$= \int_{\mathbb{R}} |g(a)|^2 \left(\int_{\mathbb{R}^+} \left| \sqrt{\lambda}\, \widehat{\Psi}(\lambda a) \right|^2 \frac{\mathrm{d}\lambda}{\lambda^2} \right) \mathrm{d}a$$

El operador N_Ψ es una isometría

$$\| N_\Psi g \|^2_{\mathrm{L}^2(\mathbb{G})} = \int_{\mathbb{R}} \int_{\mathbb{R}^+} \left| g(a)\, \overline{\widehat{\Psi}_{\frac{1}{\lambda},0}(a)} \right|^2 \frac{\mathrm{d}\lambda\, \mathrm{d}a}{\lambda^2}$$

$$= \int_{\mathbb{R}} |g(a)|^2 \left(\int_{\mathbb{R}^+} \left| \sqrt{\lambda}\, \widehat{\Psi}(\lambda a) \right|^2 \frac{\mathrm{d}\lambda}{\lambda^2} \right) \mathrm{d}a$$

$$= \int_{\mathbb{R}} |g(a)|^2 \left(\int_{\mathbb{R}^+} \frac{|\widehat{\Psi}(\lambda a)|^2}{\lambda}\, \mathrm{d}\lambda \right) \mathrm{d}a$$

El operador N_Ψ es una isometría

$$
\begin{aligned}
\| N_\Psi g \|^2_{L^2(\mathbb{G})} &= \int_{\mathbb{R}} \int_{\mathbb{R}^+} \left| g(a) \, \overline{\widehat{\Psi}_{\frac{1}{\lambda},0}(a)} \right|^2 \frac{d\lambda \, da}{\lambda^2} \\
&= \int_{\mathbb{R}} |g(a)|^2 \left(\int_{\mathbb{R}^+} \left| \sqrt{\lambda} \widehat{\Psi}(\lambda a) \right|^2 \frac{d\lambda}{\lambda^2} \right) da \\
&= \int_{\mathbb{R}} |g(a)|^2 \left(\int_{\mathbb{R}^+} \frac{|\widehat{\Psi}(\lambda a)|^2}{\lambda} \, d\lambda \right) da \\
&= \| g \|^2_{L^2(\mathbb{R})}
\end{aligned}
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&= \| g \|^2_{L^2(\mathbb{R})}
\end{aligned}
$$

El operador $\mathcal{W}_\Psi$ es isométrico

Ψ–TOC	☛	$\mathcal{W}_\Psi^*\mathcal{W}_\Psi = \mathrm{I}_\mathbb{R}$
es isometría	☛	$P_\Psi = \mathcal{W}_\Psi\mathcal{W}_\Psi^*$
	☛	$(\Psi_\gamma)_{\gamma\in\mathbb{G}}$ es un sistema de estados coherentes
	☛	$\mathcal{W}_\Psi(\mathrm{L}^2(\mathbb{R}))$ es un espacio con núcleo reproductor

La proyección ortogonal en $\mathcal{W}_\Psi(\mathrm{L}^2(\mathbb{R}))$

$$P_\Psi : \mathrm{L}^2(\mathbb{G}) \to \mathrm{L}^2(\mathbb{G})$$

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$$P_\Psi w(\mu, b) \quad = \quad \mathcal{W}_\Psi \mathcal{W}_\Psi^* w(\mu, b)$$

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$$P_\Psi : \mathrm{L}^2(\mathbb{G}) \to \mathrm{L}^2(\mathbb{G})$$

$$\begin{aligned} P_\Psi w(\mu, b) &= \mathcal{W}_\Psi \mathcal{W}_\Psi^* w(\mu, b) \\ &= \langle\, \mathcal{W}_\Psi^* w \,,\, \Psi_{\mu,b} \,\rangle \\ &= \langle\, w \,,\, \mathcal{W}_\Psi \Psi_{\mu,b} \,\rangle \end{aligned}$$

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P_\Psi w(\mu, b) &= \mathcal{W}_\Psi \mathcal{W}_\Psi^* w(\mu, b) \\
&= \langle\, \mathcal{W}_\Psi^* w \,,\ \Psi_{\mu,b} \,\rangle \\
&= \langle\, w \,,\ \mathcal{W}_\Psi \Psi_{\mu,b} \,\rangle \\
&= \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^+} w(\lambda, a)\, \overline{\langle\, \Psi_{\mu,b} \,,\ \Psi_{\lambda,a} \,\rangle}\, \frac{\mathrm{d}\lambda\, \mathrm{d}a}{\lambda^2}
\end{aligned}$$

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\end{aligned}$$

$$\boxed{P_\Psi w(\mu, b) = \int_{\mathbb{R}} \int_{\mathbb{R}^+} w(\lambda, a) \langle\, \Psi_{\lambda,a} \,,\, \Psi_{\mu,b} \,\rangle \frac{\mathrm{d}\lambda \, \mathrm{d}a}{\lambda^2}}$$

$(\Psi_\gamma)_{\gamma \in \mathbb{G}}$ como **sistema de estados coherentes**

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$$\forall\, f, g \in \mathrm{L}^2(\mathbb{R}) \;:$$

$$\langle\, f \;,\; g \,\rangle \;=\; \langle\, \mathcal{W}_\Psi f \;,\; \mathcal{W}_\Psi g \,\rangle$$

$(\Psi_\gamma)_{\gamma\in\mathbb{G}}$ como **sistema de estados coherentes**

$$\forall\, f, g \in \mathrm{L}^2(\mathbb{R}) \ :$$

$$\begin{aligned} \langle\, f\ ,\ g\,\rangle &= \langle\, \mathcal{W}_\Psi f\ ,\ \mathcal{W}_\Psi g\,\rangle \\ &= \int_{\mathbb{G}} \mathcal{W}_\Psi f(\gamma)\, \overline{\mathcal{W}_\Psi g(\gamma)}\, \mathrm{d}\nu_L(\gamma) \end{aligned}$$

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$$\langle\, f\,,\, g\,\rangle = \int_{\mathbb{G}} \langle\, f\,,\, \Psi_\gamma\,\rangle\langle\, \Psi_\gamma\,,\, g\,\rangle\,\mathrm{d}\nu_L(\gamma)$$

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$\mathcal{W}_\Psi(L^2(\mathbb{R}))$ como espacio con **núcleo reproductor**

$\mathcal{W}_{\Psi}(\mathrm{L}^2(\mathbb{R}))$ como espacio con **núcleo reproductor**

$$\forall\, w \in \mathcal{W}_{\Psi}(\mathrm{L}^2(\mathbb{R})) \quad \forall\, \gamma \in \mathbb{G}\ :$$

$$w(\gamma) = \int_{\mathbb{G}} w(\xi)\,\langle\, \Psi_{\xi}\,,\ \Psi_{\gamma}\,\rangle\, \mathrm{d}\nu_L(\xi)$$

$\mathcal{W}_{\Psi}(\mathrm{L}^2(\mathbb{R}))$ como espacio con **núcleo reproductor**

$$\forall\, w \in \mathcal{W}_{\Psi}(\mathrm{L}^2(\mathbb{R})) \quad \forall\, \gamma \in \mathbb{G} \ :$$

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Núcleo reproductor en $\mathcal{W}_{\Psi}(\mathrm{L}^2(\mathbb{R}))$:

$$K_{\gamma}(\xi) = \langle\, \Psi_{\gamma}\,,\ \Psi_{\xi}\,\rangle$$

$\mathcal{W}_\Psi(\mathrm{L}^2(\mathbb{R}))$ como espacio con **núcleo reproductor**

$$\forall\, w \in \mathcal{W}_\Psi(\mathrm{L}^2(\mathbb{R})) \quad \forall\, \gamma \in \mathbb{G} \ :$$

$$w(\gamma) = \int_{\mathbb{G}} w(\xi) \, \langle \, \Psi_\xi \, , \, \Psi_\gamma \, \rangle \, \mathrm{d}\nu_L(\xi)$$

Núcleo reproductor en $\mathcal{W}_\Psi(\mathrm{L}^2(\mathbb{R}))$:

$$K_\gamma(\xi) = \langle \, \Psi_\gamma \, , \, \Psi_\xi \, \rangle$$

$$w(\gamma) = \langle \, w \, , \, K_\gamma \, \rangle$$

Operadores de **Localización**

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- Ψ (una ondícula)
- $\sigma \in \mathrm{L}^\infty(\mathbb{G})$

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$$L_\sigma^\Psi : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$$

$$L_\sigma^\Psi = \mathcal{W}_\Psi^* M_\sigma \mathcal{W}_\Psi$$

Operadores de **Localización**

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- ☛ $\sigma \in \mathrm{L}^{\infty}(\mathbb{G})$

$$L_{\sigma}^{\Psi} : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$$

$$L_{\sigma}^{\Psi} = \mathcal{W}_{\Psi}^{*} M_{\sigma} \mathcal{W}_{\Psi}$$

$$\begin{array}{ccc} \mathrm{L}^2(\mathbb{R}) & \xrightarrow{\mathcal{W}_{\Psi}} & \mathrm{L}^2(\mathbb{G}) \\ \downarrow{\scriptstyle L_{\sigma}^{\Psi}} & & \downarrow{\scriptstyle M_{\sigma}} \\ \mathrm{L}^2(\mathbb{R}) & \xleftarrow[\mathcal{W}_{\Psi}^{*}]{} & \mathrm{L}^2(\mathbb{G}) \end{array}$$

Los operadores $\widetilde{\mathcal{W}}_{\Psi}$ y $\widetilde{P}_{\Psi}$

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$$
\begin{aligned}
\widetilde{\mathcal{W}}_\Psi : \mathrm{L}^2(\mathbb{R}) &\longrightarrow \mathcal{W}_\Psi(\mathrm{L}^2(\mathbb{R})) \\
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\end{aligned}
$$

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Los operadores $\widetilde{\mathcal{W}}_\Psi$ y $\widetilde{P}_\Psi$

$$\widetilde{\mathcal{W}}_\Psi : \mathrm{L}^2(\mathbb{R}) \longrightarrow \mathcal{W}_\Psi(\mathrm{L}^2(\mathbb{R})) \qquad \widetilde{P}_\Psi : \mathrm{L}^2(\mathbb{G}) \longrightarrow \mathcal{W}_\Psi(\mathrm{L}^2(\mathbb{R}))$$

$$\widetilde{\mathcal{W}}_\Psi f = \mathcal{W}_\Psi f \qquad \widetilde{P}_\Psi w = P_\Psi w$$

Operador de **Toeplitz-Calderón**

$$T_\sigma^\Psi : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R})$$

$$T_\sigma^\Psi = \widetilde{P}_\Psi M_\sigma \widetilde{P}_\Psi^*$$

Diagrama general

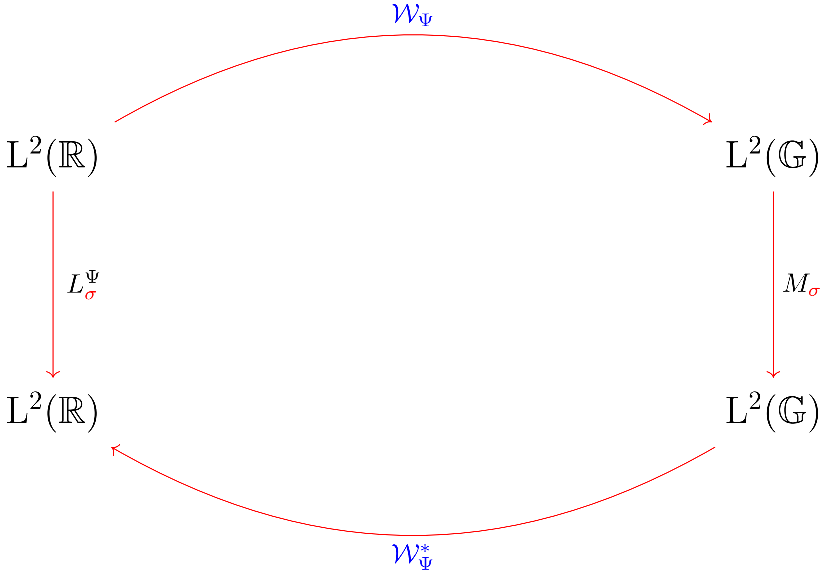
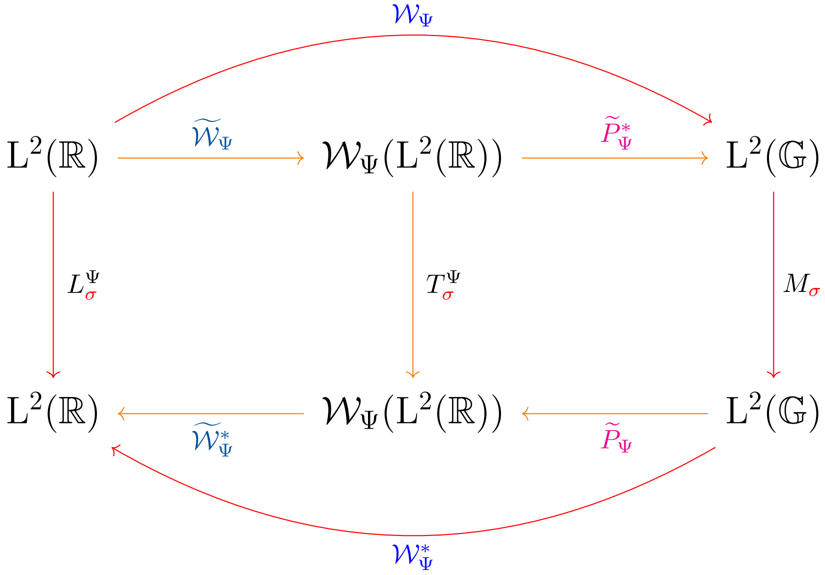


Diagrama general



Operadores de **localización** y **Toeplitz-Calderón**

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Operadores de **localización** y **Toeplitz-Calderón**

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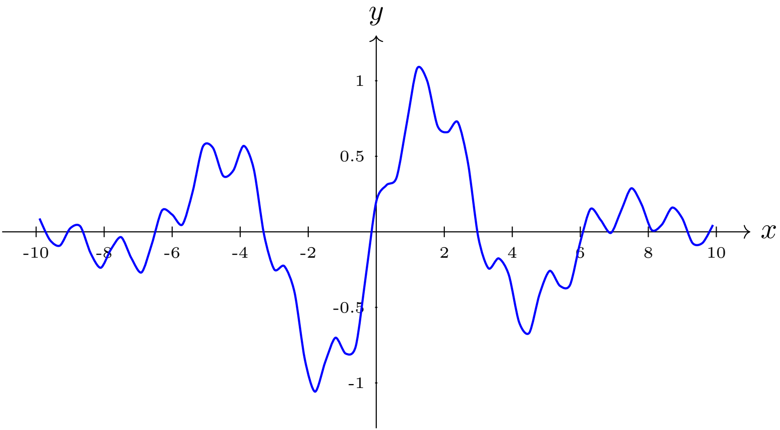
Operadores de **localización** y **Toeplitz-Calderón**

$$L_\sigma^\Psi = \mathcal{W}_\Psi^* M_\sigma \mathcal{W}_\Psi = \widetilde{\mathcal{W}}_\Psi^* \, \widetilde{P}_\Psi M_\sigma \, \widetilde{P}_\Psi^* \, \widetilde{\mathcal{W}}_\Psi$$

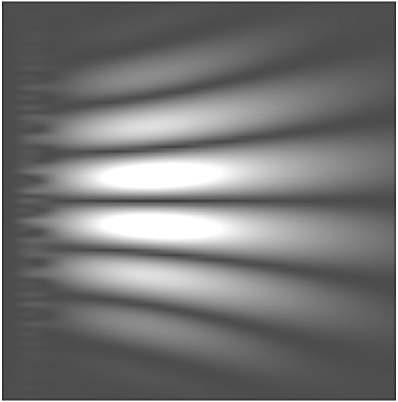
$$\boxed{L_\sigma^\Psi = \widetilde{\mathcal{W}}_\Psi^* \, T_\sigma^\Psi \, \widetilde{\mathcal{W}}_\Psi}$$

Aplicación de un operador de localización

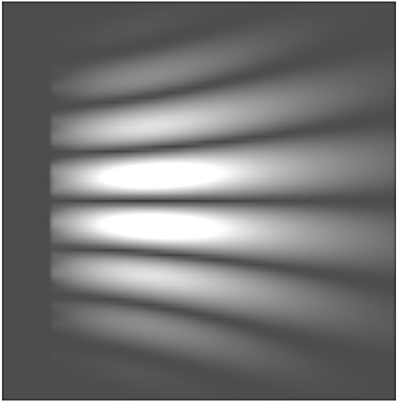
$$f(x) = e^{-x^2/32}\,\mathrm{sen}(x) + \frac{1}{5}e^{-x^2/128}\cos(5x)$$



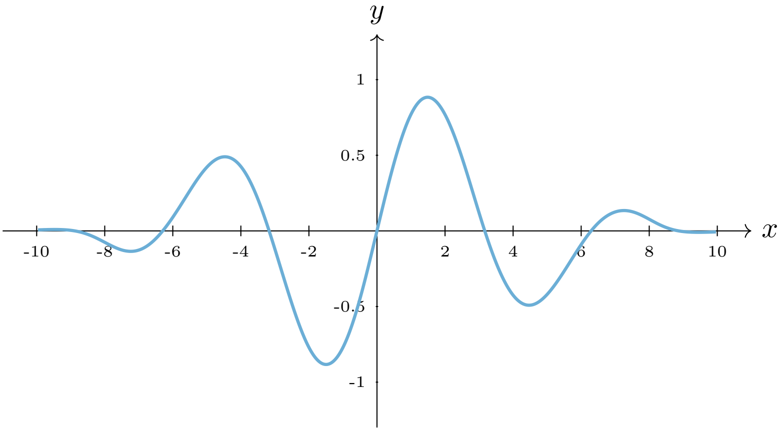
$$F(\lambda, a) = \mathcal{W}_\Psi f(\lambda, a)$$



$$g(\lambda, a) = M_{\mathbf{1}_{[3,+\infty[}} \mathcal{W}_\Psi f(\lambda, a)$$

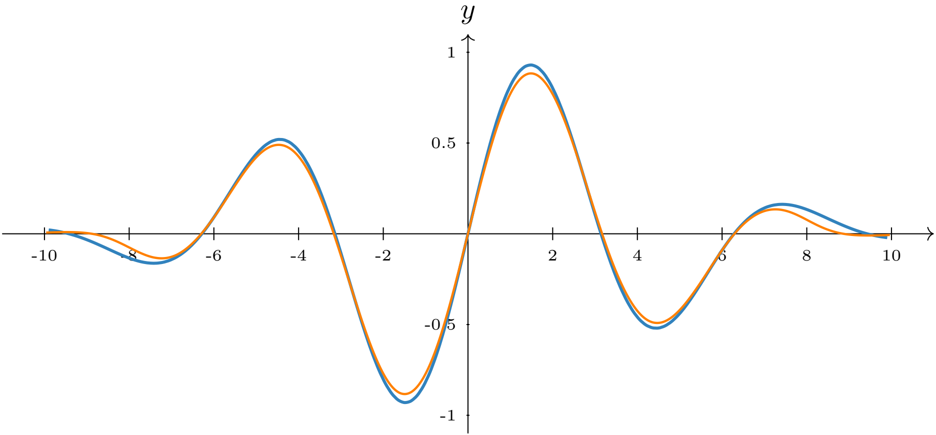


Resultado de la aplicación del operador localización



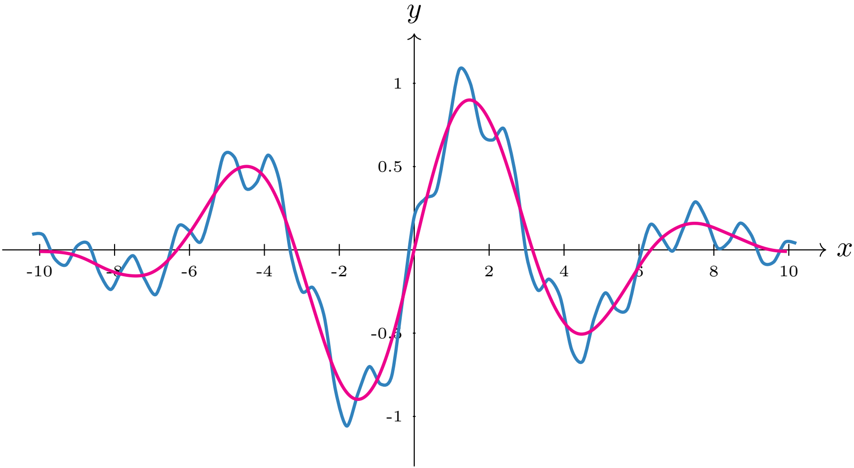
Comparación 1

$$f_1(x) = e^{-x^2/32}\,\mathrm{sen}(x) \text{ y } h(x) = L^{\Psi}_{\mathbf{1}_{]3,+\infty[}} f(x)$$

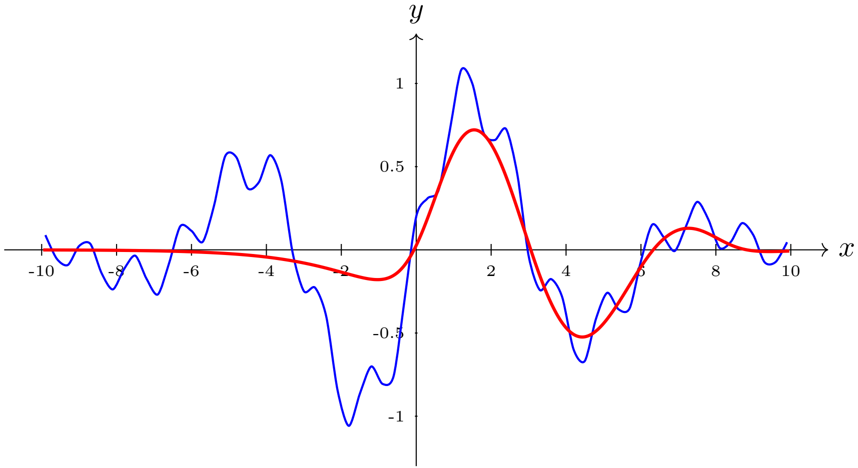


Comparación 2

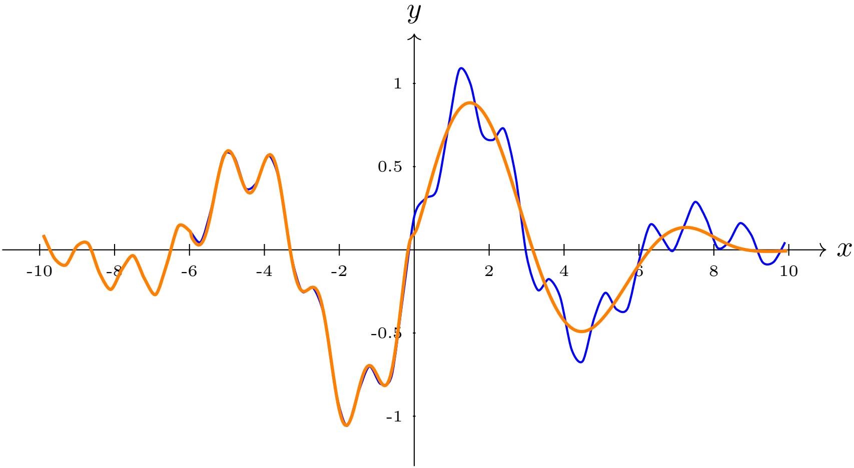
$$f(x) = e^{-x^2/32}\,\mathrm{sen}(x) + \tfrac{1}{5}e^{-x^2/128}\cos(5x) \text{ y } h(x) = L^{\Psi}_{\mathbf{1}_{]3,+\infty[}}\, f(x)$$



Otros ejemplos



Otros ejemplos



Diagonalización de los operadores de **localización**

[**Hutník** inspirado por **Vasilevski**]

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Nuevas consideraciones:

☛ $\sigma(\lambda, a) = \sigma(\lambda) \in \mathrm{L}^{\infty}(\mathbb{R})$

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$$L_\sigma^\Psi \quad = \quad \mathcal{W}_\Psi^* M_\sigma \mathcal{W}_\Psi$$

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$$\begin{aligned} L_\sigma^\Psi &= \mathcal{W}_\Psi^* M_\sigma \mathcal{W}_\Psi \\ &= \mathcal{F}^* N_\Psi^* U^* M_\sigma U N_\Psi \mathcal{F} \end{aligned}$$

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- ☛ $U\sigma = (\mathrm{I} \otimes \mathcal{F}^{-1})\sigma = \sigma$
- ☛ $UM_{\sigma} = M_{\sigma}U$

$$
\begin{aligned}
L_{\sigma}^{\Psi} &= \mathcal{W}_{\Psi}^{*} M_{\sigma} \mathcal{W}_{\Psi} \\
&= \mathcal{F}^{*} N_{\Psi}^{*} U^{*} M_{\sigma} U N_{\Psi} \mathcal{F} \\
&= \mathcal{F}^{*} (N_{\Psi}^{*} M_{\sigma} N_{\Psi}) \mathcal{F}
\end{aligned}
$$

El operador N_Ψ y su adjunto

$$N_\Psi : \mathrm{L}^2(\mathbb{R}) \longrightarrow \mathrm{L}^2(\mathbb{G})$$

$$N_\Psi g(\lambda, a) = g(a)\, \overline{\widehat{\Psi}_{\frac{1}{\lambda},0}(a)}$$

El operador N_Ψ y su adjunto

$$N_\Psi : \mathrm{L}^2(\mathbb{R}) \longrightarrow \mathrm{L}^2(\mathbb{G})$$

$$N_\Psi g(\lambda, a) = g(a)\,\overline{\widehat{\Psi}_{\frac{1}{\lambda},0}(a)} = \sqrt{\lambda}\, g(a)\,\overline{\widehat{\Psi}(\lambda a)}$$

El operador N_{Ψ} y su adjunto

$$N_{\Psi} : \mathrm{L}^2(\mathbb{R}) \longrightarrow \mathrm{L}^2(\mathbb{G})$$

$$N_{\Psi} g(\lambda, a) = g(a)\,\overline{\widehat{\Psi}_{\frac{1}{\lambda},0}(a)} = \sqrt{\lambda}\,g(a)\,\overline{\widehat{\Psi}(\lambda a)}$$

$$N_{\Psi}^{*} : \mathrm{L}^2(\mathbb{G}) \longrightarrow \mathrm{L}^2(\mathbb{R})$$

$$N_{\Psi}^{*} w(x) = \int_{\mathbb{R}^+} \frac{w(\lambda, x)\widehat{\Psi}(\lambda x)}{\lambda\sqrt{\lambda}}\,\mathrm{d}\lambda$$

El operador $N_\Psi^* M_\sigma \, N_\Psi : \mathrm{L}^2(\mathbb{R}) \longrightarrow \mathrm{L}^2(\mathbb{R})$

$$N_\Psi^* M_\sigma \, N_\Psi g(x) \quad =$$

El operador $N_\Psi^* M_\sigma \, N_\Psi : \mathrm{L}^2(\mathbb{R}) \longrightarrow \mathrm{L}^2(\mathbb{R})$

$$N_\Psi^* M_\sigma \, N_\Psi g(x) \quad = \quad \int\limits_{\mathbb{R}^+} \frac{M_\sigma \, N_\Psi g(\lambda, x) \widehat{\Psi}(\lambda x)}{\lambda \sqrt{\lambda}} \, \mathrm{d}\lambda$$

El operador $N_{\Psi}^{*} M_{\sigma} N_{\Psi} : \mathrm{L}^2(\mathbb{R}) \longrightarrow \mathrm{L}^2(\mathbb{R})$

$$
\begin{aligned}
N_{\Psi}^{*} M_{\sigma} N_{\Psi} g(x) &= \int\limits_{\mathbb{R}^+} \frac{M_{\sigma} N_{\Psi} g(\lambda, x) \widehat{\Psi}(\lambda x)}{\lambda\sqrt{\lambda}} \, \mathrm{d}\lambda \\
&= \int\limits_{\mathbb{R}^+} \frac{\sigma(\lambda)\sqrt{\lambda}\, g(x)\, \overline{\widehat{\Psi}(\lambda x)}\, \widehat{\Psi}(\lambda x)}{\lambda\sqrt{\lambda}} \, \mathrm{d}\lambda
\end{aligned}
$$

El operador $N_\Psi^* M_\sigma N_\Psi : \mathrm{L}^2(\mathbb{R}) \longrightarrow \mathrm{L}^2(\mathbb{R})$

$$\begin{aligned}
N_\Psi^* M_\sigma N_\Psi g(x) &= \int_{\mathbb{R}^+} \frac{M_\sigma N_\Psi g(\lambda, x) \widehat{\Psi}(\lambda x)}{\lambda\sqrt{\lambda}} \,\mathrm{d}\lambda \\
&= \int_{\mathbb{R}^+} \frac{\sigma(\lambda)\sqrt{\lambda}\, g(x)\, \overline{\widehat{\Psi}(\lambda x)}\, \widehat{\Psi}(\lambda x)}{\lambda\sqrt{\lambda}} \,\mathrm{d}\lambda \\
&= \underbrace{\left(\int_{\mathbb{R}^+} \sigma(\lambda) \frac{|\widehat{\Psi}(\lambda x)|^2}{\lambda} \,\mathrm{d}\lambda \right)}_{\gamma_\sigma^\Psi} g(x)
\end{aligned}$$

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$$
\begin{aligned}
N_{\Psi}^{*} M_{\sigma} N_{\Psi} g(x) &= \int_{\mathbb{R}^+} \frac{M_{\sigma} N_{\Psi} g(\lambda, x) \widehat{\Psi}(\lambda x)}{\lambda\sqrt{\lambda}} \, \mathrm{d}\lambda \\
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&= \underbrace{\left(\int_{\mathbb{R}^+} \sigma(\lambda) \frac{|\widehat{\Psi}(\lambda x)|^2}{\lambda} \, \mathrm{d}\lambda \right)}_{\gamma_{\sigma}^{\Psi}} g(x) \\
&= M_{\gamma_{\sigma}^{\Psi}} \, g(x)
\end{aligned}
$$

... de regreso a la diagonalización:

$$L_{\sigma}^{\Psi} = \mathcal{F}^* N_{\Psi}^* M_{\sigma} N_{\Psi} \mathcal{F}$$

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$$L_{\sigma}^{\Psi} = \mathcal{F}^* \, N_{\Psi}^* M_{\sigma} \, N_{\Psi} \mathcal{F}$$

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... de regreso a la diagonalización:

$$L_{\sigma}^{\Psi} = \mathcal{F}^{*} N_{\Psi}^{*} M_{\sigma} N_{\Psi} \mathcal{F}$$

$$\boxed{\mathcal{F}\, L_{\sigma}^{\Psi}\, \mathcal{F}^{*} = M_{\gamma_{\sigma}^{\Psi}}}$$

por lo tanto:

$$\| L_{\sigma}^{\Psi} \| = \| \gamma_{\sigma}^{\Psi} \|_{\infty} \quad \text{y} \quad \mathrm{Sp}(L_{\sigma}^{\Psi}) = \mathrm{clos}(\gamma_{\sigma}^{\Psi}(\mathbb{R}))$$

... de regreso a la diagonalización:

$$L_\sigma^\Psi = \mathcal{F}^* N_\Psi^* M_\sigma N_\Psi \mathcal{F}$$

$$\boxed{\mathcal{F} L_\sigma^\Psi \mathcal{F}^* = M_{\gamma_\sigma^\Psi}}$$

por lo tanto:

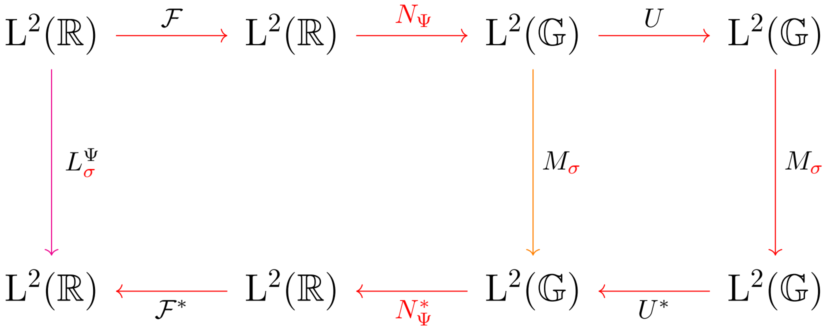
$$\| L_\sigma^\Psi \| = \| \gamma_\sigma^\Psi \|_\infty \quad \text{y} \quad \mathrm{Sp}(L_\sigma^\Psi) = \mathrm{clos}(\gamma_\sigma^\Psi(\mathbb{R}))$$

Hutník O.; Hutníková M.(2011): *On Toeplitz localization operators.*

Diagrama general para la diagonalización

$$
\begin{array}{ccccccc}
\mathrm{L}^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & \mathrm{L}^2(\mathbb{R}) & \xrightarrow{N_\Psi} & \mathrm{L}^2(\mathbb{G}) & \xrightarrow{U} & \mathrm{L}^2(\mathbb{G}) \\
\Big\downarrow L_\sigma^\Psi & & & & & & \Big\downarrow M_\sigma \\
\mathrm{L}^2(\mathbb{R}) & \xleftarrow[\mathcal{F}^*]{} & \mathrm{L}^2(\mathbb{R}) & \xleftarrow[N_\Psi^*]{} & \mathrm{L}^2(\mathbb{G}) & \xleftarrow[U^*]{} & \mathrm{L}^2(\mathbb{G})
\end{array}
$$

Diagrama general para la diagonalización



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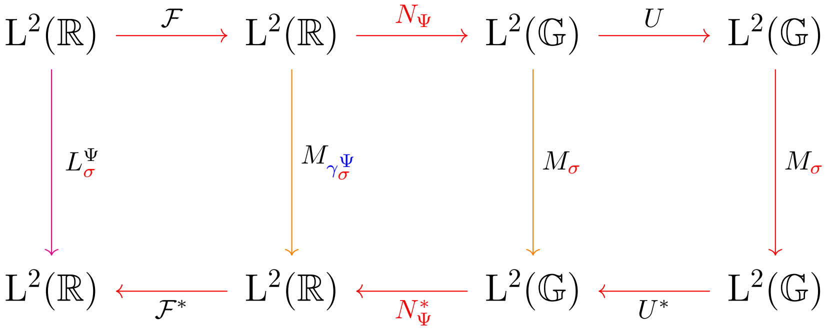

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Esmeral, K.; Maximenko, E. (2016): *Radial Toeplitz operators on the Fock space and square-root-slowly oscillating sequences.*

Densidad de las funciones espectrales

Teorema 1. Si $\mathcal{F}K_\Psi(\xi) \neq 0$ para todo $\xi \in \mathbb{R}$ (condición de Wiener), entonces el conjunto

$$\mathcal{G}_\Psi = \{\, \sigma * K_\Psi \; : \; \sigma \in \mathrm{L}^\infty(\mathbb{R}) \,\}$$

es un subespacio denso de $\mathcal{C}_u(\mathbb{R})$.

El álgebra C* generada por los operadores de localización cuyo símbolo generador depende sólo del parámetro de escalamiento

☛ $\widehat{\Psi}_k(\xi) = \sqrt{2|\xi|}\,\ell_k(2|\xi|)$

Teorema 2. El álgebra C* generada por los operadores de localización relativos a estas ondículas cuyos símbolos generadores dependen sólo del parámetro de escalamiento es isométricamente isomorfa a $\mathcal{C}_u(\mathbb{R})$.

Hutník, O.; Maximenko, E.; Miškova, A. (2016), *Toeplitz localization operators: spectral functions density.*

Referencias

Hutník, O.; Maximenko, E.; Miškova, A. (2016), *Toeplitz localization operators: spectral functions density.* Complex Anal. Oper. Theory, 10:8, 1757–1774. DOI: 10.1007/s11785-016-0564-1

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Hutník O.; Hutníková M.(2011): *On Toeplitz localization operators.* Arch. Math. 97, 333–344 DOI: 10.1007/s00013-011-0307-5.

¡Gracias!

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Спасибо!