Examples of reproducing kernel Hilbert spaces with translation-invariant operators

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Outline

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A general problem (we cannot solve it)

Let

X be a set,

G be a locally compact group,

$\alpha\colon G \to \operatorname{Sym}(X)$ be a group action,

H be a reproducing kernel Hilbert space over X,

$(\rho_{\alpha,H}(g))_{g\in G}$, $\quad \rho_{\alpha,H}(g)f := f\circ\alpha(g^{-1})$ be a unitary representation.

Problem: describe the W*-algebra defined as the centralizer of $\rho_{\alpha,H}$,

$$\mathcal{V} := \rho'_{\alpha,H} := \Big\{S\in\mathcal{B}(H)\colon \quad \forall g\in G \quad S\,\rho_{\alpha,H}(g) = \rho_{\alpha,H}(g)\,S\Big\}.$$

A general idea (we are not prepared enough to implement it)

Apply the Fourier transform to the reproducing kernel along the orbits of the group action:

$$\int_G K_z(\alpha(g)(w))\overline{\psi(g)}\,\mathrm{d}\nu_G(g), \qquad \psi \in \text{irreducible representations of } G.$$

We hope that the obtained operator-valued function is useful to describe $\mathcal{V}$.

Our assumptions

- $X = G \times Y$,
- G is an Abelian locally compact group, metrizable, and σ-compact,
- Y is a σ-finite measure space,
- $\alpha(g)\colon\ (x,y) \mapsto (g+x,y)$,
- $L^2(G \times Y)$ is separable,
- $\rho_{G\times Y}$ acts in $L^2(G\times Y)$ by $(\rho_{G\times Y}(a)f)(x,y) := f(x-a,y)$,
- $H \le L^2(G\times Y)$, H is invariant under $\rho_{G\times Y}$,
- H is an RKHS; we denote the RK be $(K_{(x,y)})_{x\in G,\ y\in Y}$,
- $\forall y \in Y \quad \sup_{v\in Y} \int_G |K_{(0,y)}(u,v)|\,\mathrm{d}\nu(u) < +\infty$.

Criterion that H is shift-invariant

$P :=$ the orthogonal projection in $L^2(G \times Y)$ whose image is H.

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Proposition

The following conditions are equivalent.

(a) $\rho_{G\times Y}(H) \subseteq H$ for every a in G.

(b) $P\rho_{G\times Y}(a) = \rho_{G\times Y}(a)P$ for every a in G,

(c) $K_{x,y}(u,v) = K_{0,y}(u-x,v)$ for every x, y in G and every y, v in Y.

(d) $\rho_{G\times Y}(a)K_{x,y} = K_{a+x,y}$ for every a, x in G and every y in Y.

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(d) $\rho_{G\times Y}(a)K_{x,y} = K_{a+x,y}$ for every a, x in G and every y in Y.

Let $\rho_H : H \to H$ be the compression of $\rho_{G\times Y}$.

Decomposition of H

$$\widehat{P} := (F \otimes I) P (F \otimes I)^*, \qquad \widehat{H} := (F \otimes I)(H).$$

$$\widehat{P} = \int_{\widehat{G}}^{\oplus} \widehat{P}_\xi \, \mathrm{d}\widehat{\nu}(\xi).$$

For each ξ in $\widehat{G}$,

$$\widehat{H}_\xi := \widehat{P}_\xi(L^2(Y)).$$

$$\Omega := \{\xi \in \widehat{G} \colon \ \dim(\widehat{H}_\xi) > 0\}.$$

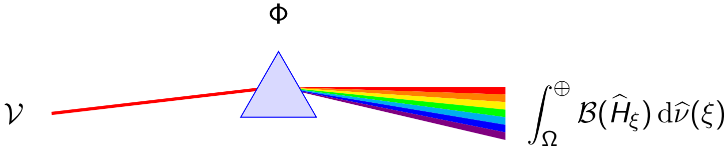
$$\widehat{H} = \int_{\Omega}^{\oplus} \widehat{H}_\xi \, \mathrm{d}\widehat{\nu}(\xi).$$

Decomposition of $\mathcal{V} = \rho'_H$

Let $\Phi\colon H \to \widehat{H}$ be the compression of $F \otimes I$.

Theorem

$$\Phi \mathcal{V} \Phi^* = \int_\Omega^\oplus \mathcal{B}(\widehat{H}_\xi)\,\mathrm{d}\widehat{\nu}(\xi).$$



Constructive description of the fibers $\widehat{H}_\xi$

$$L_{\cdot,y} := (F \otimes I) K_{0,y}, \qquad \text{i.e.,} \qquad L_{\xi,y}(v) := \int_G K_{(0,y)}(u,v)\,\overline{\xi(u)}\,\mathrm{d}\nu(u).$$

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Theorem

For every ξ in Ω, the family $(L_{\xi,y})_{y\in Y}$ is the reproducing kernel of $\widehat{H}_\xi$.

Idea of the proof: convolution theorem + Fubini + Moore–Aronszajn theorem.

Corollary:

$$\dim(\widehat{H}_\xi) = \int_Y L_{\xi,y}(y)\,\mathrm{d}\nu(y).$$

Constructive criterion for the commutativity of $\mathcal{V}$

Theorem

The following conditions are equivalent.

(a) $\mathcal{V}$ is commutative.

(b) $\dim(\widehat{H}_\xi) = 1$ for every ξ in Ω.

(c) $\displaystyle\int_Y L_{\xi,y}(y)\,\mathrm{d}\lambda(y) = 1$ for every ξ in Ω,

(d) $|L_{\xi,y}(v)|^2 = L_{\xi,y}(y)L_{\xi,v}(v)$ for every ξ in Ω and every y, v in Y.

(e) There exists a family $(q_\xi)_{\xi\in\Omega}$ in $L^2(Y)$ such that the function $(\xi, v) \mapsto q_\xi(v)$ is measurable, the function q_ξ forms an orthonormal basis of $\widehat{H}_\xi$, and

$$L_{\xi,y}(v) = \overline{q_\xi(y)}q_\xi(v) \qquad (\xi \in \Omega,\ y, v \in Y).$$

The case of finite-dimensional fibers

Suppose that

$$\forall \xi \in \Omega \qquad d_\xi := \dim(\widehat{H}_\xi) < +\infty.$$

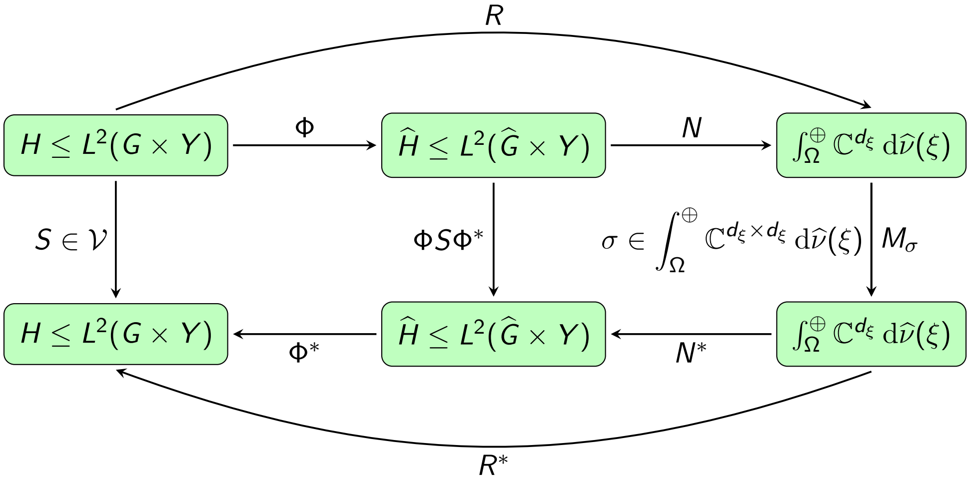
Let $(q_{j,\xi})_{j\in\mathbb{N},\xi\in\Omega}$ be a measurable basis family for the spaces $\widehat{H}_\xi$.

$$L_{\xi,y}(v) = \sum_{j=1}^{d_\xi} \overline{q_{j,\xi}(y)}\, q_{j,\xi}(v).$$

Then

$$\Phi H = \widehat{H} = \int_\Omega^\oplus \widehat{H}_\xi \,\mathrm{d}\widehat{\nu}(\xi) \cong \int_\Omega^\oplus \mathbb{C}^{d_\xi} \,\mathrm{d}\widehat{\nu}(\xi).$$

From translation-invariant operators to matrix families



Berezin transform of translation-invariant operators

Corollary

If $S \in \mathcal{V}$, $RSR^* = M_\sigma$. Then

$$\operatorname{Ber}(S)(x,y) = \frac{\displaystyle\int_\Omega \sigma(\xi) L_{\xi,y}(y)\,\mathrm{d}\widehat{\nu}(\xi)}{\displaystyle\int_\Omega L_{\xi,y}(y)\,\mathrm{d}\widehat{\nu}(\xi)} \qquad (x \in G,\ \ y \in Y).$$

In particular, $\operatorname{Ber}(S)(x,y)$ does not depend on x.

Matrix families corresponding to Toeplitz operators with translation-invariant generating symbols

Corollary

Let $\psi \in L^{\infty}(Y)$,

$$\varphi(x, y) = \psi(y).$$

Then $T_{\varphi} \in \mathcal{V}$, $RT_{\varphi}R^{*} = M_{\gamma_{\psi}}$,

$$\gamma_{\psi}(\xi) := \left[\int_{Y} \psi(v)\overline{q_{j,\xi}(v)}q_{k,\xi}(v)\,\mathrm{d}\lambda(v)\right]_{j,k=1}^{d_{\xi}}.$$

Simple examples: three group actions, several spaces

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	vertical	radial	angular
analytic	?	?	?
harmonic	?	?	?
n-analytic	?	?	?
(*n*)-analytic	?	?	?
n-harmonic	?	?	?
wavelet	?		

Outline

Upper half-plane Π

$\mu_n :=$ the Lebesgue measure in $\mathbb{R}^n$.

$\Pi = G \times Y$, $G = \mathbb{R}$, $\widehat{G} \cong \mathbb{R}$, $Y = \mathbb{R}_+ = (0, +\infty)$.

Pairing between $G = \mathbb{R}$ and $\widehat{G} \cong \mathbb{R}$:

$$E(x, \xi) = \mathrm{e}^{\mathrm{i}\, x\xi}.$$

Measures on $G = \mathbb{R}$, $\widehat{G}$, and Y:

$$\nu = \widehat{\nu} = \frac{1}{\sqrt{2\pi}}\,\mu_1, \qquad \lambda = \sqrt{2\pi}\mu_1, \qquad \nu \times \lambda = \mu_2.$$

$H = L^2_a(\Pi)$

$$K_z(w) = -\frac{1}{\pi(w-\overline{z})^2}.$$

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After identifying Π with $\mathbb{R} \times \mathbb{R}_+$,

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$$K_{x,y}(u,v) = -\frac{1}{\pi((u-x)+\mathrm{i}(v+y))^2}.$$

$$L_{\xi,y}(v) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} K_{0,y}(u,v)\,\mathrm{e}^{-\mathrm{i}\,u\xi}\,\mathrm{d}u = \sqrt{\frac{2}{\pi}}\,\xi\,\mathrm{e}^{-\xi(y+v)}\,1_{\mathbb{R}_+}(\xi).$$

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The spectral functions of vertical Toeplitz operators:

$$\gamma_\sigma(\xi) = \int_\Omega \sigma(v) |q_\xi(v)|^2 \,\mathrm{d}\lambda(v) = 2\xi \int_{\mathbb{R}_+} \sigma(v)\, \mathrm{e}^{-2\xi v}\, \mathrm{d}v \qquad (\xi > 0).$$

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Vasilevski (1999).

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W*-algebra of vertical operators: $\mathcal{V} \cong L^\infty(\mathbb{R}_+)$.

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C*-algebra $\mathcal{VT}$ generated by vertical Toeplitz operators:

bounded log-oscillating functions on $\mathbb{R}_+$.

Herrera-Yañez, Maximenko, Vasilevski (2013).

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In this example, $\mathcal{VT}$ is weakly dense in $\mathcal{V}$.

$H = L^2_{\mathrm{harm}}(\Pi)$

$$L^2_{\mathrm{harm}}(\Pi) = L^2_{\mathrm{a}}(\Pi) \oplus \overline{L^2_{\mathrm{a}}(\Pi)}, \qquad K_z(w) = -\frac{1}{\pi(w-\overline{z})^2} - \frac{1}{\pi(\overline{w}-z)^2}.$$

$$L_{\xi,y}(v) = \sqrt{\frac{2}{\pi}}\,|\xi|\,\mathrm{e}^{-|\xi|(y+v)} \qquad (\xi \in \mathbb{R}).$$

Conclusions: $\mathcal{V}$ is commutative, $\Omega = \mathbb{R} \setminus \{0\}$,

$$q_\xi(v) = \left(\frac{2}{\pi}\right)^{1/4} \sqrt{|\xi|}\,\mathrm{e}^{-2|\xi| v}.$$

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$\mathcal{VT}$ is not weakly dense in $\mathcal{V}$.

$H = L^2_{n\text{-analytic}}(\Pi)$, polyanalytic Bergman space

n-analytic functions are defined by the equation

$$\frac{\partial^n f}{\partial \overline{z}^n} = 0.$$

$$K_z(w) = \frac{n\,(-1)^n}{\pi}\,\frac{(z-\overline{w})^{n-1}}{(w-\overline{z})^{n+1}}\,P_{n-1}^{(0,1)}\left(2\,\frac{|w-z|^2}{|w-\overline{z}|^2}-1\right).$$

Pessoa (2013), Leal-Pacheco, Maximenko, Ramos-Vazquez (2021).

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Pessoa (2013), Leal-Pacheco, Maximenko, Ramos-Vazquez (2021).

In this example, we are unable to compute directly the Fourier transform of K.

$H = L^2_{n\text{-analytic}}(\Pi)$, polyanalytic Bergman space

Vasilevski (1999).

Applied the Fourier transform to the equation $\frac{\partial^n f}{\partial \overline{z}^n} = 0$,

$$L_{\xi,y}(v) = 1_{\mathbb{R}_+}(\xi)\sqrt{\frac{2}{\pi}}\,\xi\,\mathrm{e}^{-\xi(y+v)}\sum_{m=1}^{n} L_{m-1}(2\xi y)L_{m-1}(2\xi v).$$

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Conclusion: $\Omega = \mathbb{R}_+$, $d_\xi = n$, $\mathcal{V}$ is not commutative,

$$q_{j,\xi}(v) = (2/\pi)^{1/4}\,\sqrt{\xi}\,\mathrm{e}^{-\xi v}\,L_{j-1}(2\xi v) \qquad (j = 1,\ldots,n,\ \xi > 0,\ v > 0).$$

$H = L^2_{n\text{-analytic}}(\Pi)$, polyanalytic Bergman space

$$\mathcal{V} \cong \int_{\mathbb{R}_+}^{\oplus} \mathbb{C}^{n\times n}\,\mathrm{d}\widehat{\nu}(\xi) \cong L^\infty(\mathbb{R}_+, \mathbb{C}^{n\times n}).$$

Vertical Toeplitz operators can be transformed into matrix-functions:

$$\gamma_\sigma(\xi) := \left[\int_Y \sigma(v)\overline{q_{j,\xi}(v)}q_{k,\xi}(v)\,\mathrm{d}\lambda(v)\right]_{j,k=1}^{d_\xi}.$$

Ramírez-Ortega, Sánchez–Nungaray (2015).

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Hutník (2011), Ramírez-Ortega, Sánchez–Nungaray (2015).

Vasilevski ladder

decomposition of $L^2(\Pi)$ into true-analytic and true-anti-analytic spaces

$L^2_{(3)\text{-anti-analytic}}(\Pi)$	$L^2_{(3)\text{-analytic}}(\Pi)$
$L^2_{(2)\text{-anti-analytic}}(\Pi)$	$L^2_{(2)\text{-analytic}}(\Pi)$
$L^2_{(1)\text{-anti-analytic}}(\Pi)$	$L^2_{(1)\text{-analytic}}(\Pi)$
$\Omega = \mathbb{R}_-$	$\Omega = \mathbb{R}_+$

ξ

Wavelet spaces

Let $\psi \in L^2(\mathbb{R})$ be a mother wavelet satisfying the admissibility condition:

$$\int_{\mathbb{R}_+} |(F\psi)(t\xi)|^2 \frac{\mathrm{d}t}{t} = 1 \qquad (\xi \in \mathbb{R} \setminus \{0\}), \qquad (F\psi)(0) = 0.$$

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$W_\psi \colon L^2(\mathbb{R}) \to L^2(G \times Y)$ by

$$(W_\psi f)(x,y) := \langle f, \psi_{x,y} \rangle_{L^2(\mathbb{R})}.$$

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The wavelet space $H := W_\psi(L^2(\mathbb{R}))$.

Wavelet spaces

$$K_{x,y}(u,v) = \langle \psi_{u,v}, \psi_{x,y} \rangle_{L^2(\mathbb{R})} = \langle \psi_{u-x,v}, \psi_{0,y} \rangle_{L^2(\mathbb{R})}.$$

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$$L_{\xi,y}(v) = \sqrt{yv}\,(F\psi)(y\xi)\,\overline{(F\psi)(v\xi)}.$$

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$$L_{\xi,y}(v) = \sqrt{yv}\,(F\psi)(y\xi)\,\overline{(F\psi)(v\xi)}.$$

Conclusion: $\Omega = \mathbb{R}$, $d_\xi = 1$, $\mathcal{V}$ is commutative, and

$$q_\xi(v) = \sqrt{v}\,\overline{(F\psi)(v\xi)}.$$

Hutník, Hutníková (2011).

Similar examples

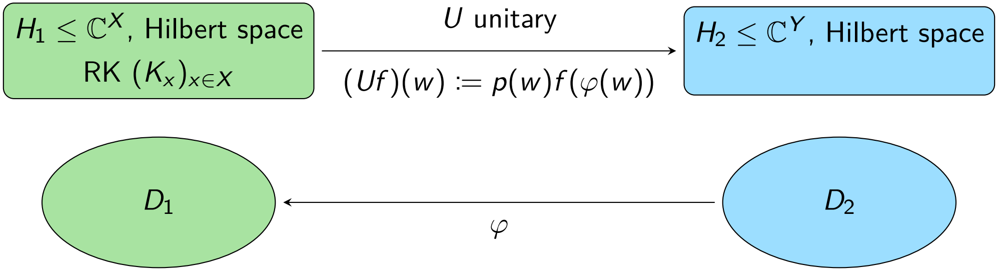
Vertical operators in the poly-harmonic spaces.

Vertical operators in the space associated to the continuous Stockwell transform.

spaces	vertical operators
analytic	$\Omega = \mathbb{R}_+,\ d_\xi = 1$
harmonic	$\Omega = \mathbb{R},\ d_\xi = 1$
n-analytic	$\Omega = \mathbb{R}_+,\ d_\xi = n$
(n)-analytic	$\Omega = \mathbb{R}_+,\ d_\xi = 1$
n-harmonic	$\Omega = \mathbb{R},\ d_\xi = n$
(n)-harmonic	$\Omega = \mathbb{R},\ d_\xi = 1$
wavelet	$\Omega = \mathbb{R}_+,\ d_\xi = 1$

Outline

Transformation of RK by a weighted change of variable



Then the following function is the RK of H_2:

$$L_z(w) = \overline{p(z)}\, p(w)\, K_{\varphi(z)}(\varphi(w)).$$

$H_1 = L^2_a(\mathbb{D})$, the analytic Bergman space

$$K_z^{H_1}(w) = \frac{1}{\pi(1-\overline{z}w)^2}.$$

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Pairing: $E(u + 2\pi\mathbb{Z}, \xi) = \mathrm{e}^{\mathrm{i}\, u\xi}$.

$Y = [0, 1)$, $\mathrm{d}\lambda(v) = v\,\mathrm{d}v$.

Transformation of the kernel

Define $\varphi\colon G\times Y\to\mathbb{D}$ and $p\colon G\times Y\to\mathbb{C}$ by

$$\varphi(u,v)=v\,\mathrm{e}^{\mathrm{i}\,u},\qquad p(u,v)=\sqrt{2\pi}.$$

U converts H_1 into a certain RKHS H over $G\times Y$, with reproducing kernel

$$K_{x,y}(u,v)=\overline{p(x,y)}\,K^{H_1}_{\varphi(x,y)}(\varphi(u,v))\,p(u,v)=\frac{2}{(1-yv\,\mathrm{e}^{\mathrm{i}(u-x)})^2}.$$

Moreover, U intertwines the rotation operators acting in H_1 with the horizontal translations acting in H.

Computation of L

$K_{0,y}(\cdot\,, v)$ decomposes into the Fourier series:

$$K_{0,y}(u,v) = \frac{2}{(1 - yv\,\mathrm{e}^{\mathrm{i}(u-x)})^2} = \sum_{\xi=0}^{\infty} 2(\xi+1)\,(yv)^{\xi}\;\mathrm{e}^{\mathrm{i}\,\xi u}\,.$$

The Fourier coefficients are

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Conclusion: $\Omega = \mathbb{N}_0$, $d_\xi = 1$, $\mathcal{V}$ is commutative, $q_\xi(v) = \sqrt{2(\xi+1)}\,v^{\xi}$.

The eigenvalues of radial Toeplitz operators in $L_a^2(\mathbb{D})$

$$q_\xi(v) = \sqrt{2(\xi+1)}\, v^\xi.$$

$$\gamma_\sigma(\xi) = \int_0^1 \sigma(v)\,|q_\xi(v)|^2\,\mathrm{d}\lambda(v) = (\xi+1)\int_0^1 \sigma(\sqrt{r})\, r^\xi\,\mathrm{d}r \qquad (\xi\in\mathbb{N}_0).$$

Radial operators in $H_1 = L^2_{\text{harm}}(\mathbb{D})$

$$K_z^{H_1}(w) = \frac{1}{\pi(1-\overline{z}w)^2} + \frac{1}{\pi(1-\overline{w}z)^2} - 1.$$

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After passing to the polar coordinates and computing the Fourier coefficients, we have

$$L_{\xi,y}(v) = 2(|\xi|+1)(yv)^{|\xi|} \qquad (\xi \in \mathbb{Z},\ y,v \in [0,1)).$$

$$\Omega = \mathbb{Z}, \qquad d_\xi = 1, \quad q_\xi(v) = \sqrt{2(|\xi|+1)}\, v^{|\xi|}, \quad \gamma_\sigma(\xi) = (|\xi|+1)\int_0^1 \sigma(\sqrt{r})\, r^{|\xi|}\,\mathrm{d}r.$$

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Loaiza, Lozano (2013).

Eigenvalues of radial Toeplitz operators in $L^2_{\mathrm{harm}}(\mathbb{D})$

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$\mathcal{VT} \cong$ bounded even log-oscillating sequences.

$\mathcal{V} \cong \ell^\infty(\mathbb{Z})$,

$\mathcal{VT}$ is not weakly dense in $\mathcal{V}$.

Radial operators in other spaces

- $L^2_{n\text{-analitic}}(\mathbb{D})$,
- $L^2_{(n)\text{-analitic}}(\mathbb{D})$,
- $L^2_{n\text{-harmonic}}(\mathbb{D})$,
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The direct computation of L is complicated,

but we know the final answers thanks to the orthonormal basis.

spaces	radial operators
analytic	$\Omega = \mathbb{N}_0,\ d_\xi = 1$
harmonic	$\Omega = \mathbb{Z},\ d_\xi = 1$
n-analytic	$\Omega = \{-n+1, -n+2, \ldots\},\ d_\xi = 1, 2, \ldots, n, n, \ldots$
(n)-analytic	$\Omega = \{-n+1, -n+2, \ldots\},\ d_\xi = 1$
n-harmonic	$\Omega = \mathbb{Z},\ d_\xi = n$
(n)-harmonic	$\Omega = \mathbb{Z},\ d_\xi = 1$

Outline

Weighted dilations in $L^2_{\mathsf{a}}(\Pi)$ and angular operators

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An operator S is angular if it commutes with $D(h)$ for every $h > 0$.

Passing from $H_1 = L^2_{\mathrm{a}}(\Pi)$ to $H \le L^2(G \times Y)$

Let $G = \mathbb{R}$, $Y = (0, \pi)$,

$$\nu = \widehat{\nu} = \frac{1}{\sqrt{2\pi}} \mu_1, \qquad E(x, \xi) = \mathrm{e}^{\mathrm{i}\, x \xi},$$

λ be the Lebesgue measure on $(0, \pi)$.

Define $\varphi\colon G \times Y \to \Pi$, $p\colon G \times Y \to \mathbb{C}$, and $U\colon H_1 \to L^2(G \times Y)$ by

$$\varphi(u, v) := \mathrm{e}^{u + \mathrm{i}\, v}, \quad p(u, v) := (2\pi)^{1/4}\, \mathrm{e}^{u + \mathrm{i}\, v},$$

$$(Uf)(u, v) = (2\pi)^{1/4}\, \mathrm{e}^{u + \mathrm{i}\, v}\, f(\mathrm{e}^{u + \mathrm{i}\, v}).$$

U is a linear isometry, $H := U(H_1)$.

Passing from $H_1 = L^2_a(\Pi)$ to $H \leq L^2(G \times Y)$

$$(Uf)(u,v) = (2\pi)^{1/4}\, e^{u+i\,v}\, f(e^{u+i\,v}).$$

U intertwines the dilations $D(h)$ acting in $\mathcal{H}_1$
with the horizontal translations acting in H:

$$UD_{e^a}U^* = \rho_H(a).$$

Compute K and L

The space $H := U(H_1)$, $H \leq L^2(G \times Y)$, has reproducing kernel

$$K_{x,y}(u,v) = -\sqrt{\frac{2}{\pi}}\frac{e^{x-\mathrm{i}\,y}\;e^{u+\mathrm{i}\,v}}{(e^{u+\mathrm{i}\,v} - e^{x-\mathrm{i}\,y})^2} = -\sqrt{\frac{2}{\pi}}\frac{1}{4\left(\sinh\frac{u-x+\mathrm{i}(v+y)}{2}\right)^2}.$$

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Its Fourier transform is

$$L_{\xi,y}(v) = \frac{2\xi\ \mathrm{e}^{-\xi(y+v)}}{1-\mathrm{e}^{-2\pi\xi}} = \overline{q_\xi(y)}\, q_\xi(v), \qquad q_\xi(v) = \sqrt{\frac{2\xi}{1-\mathrm{e}^{-2\pi\xi}}}\ \mathrm{e}^{-\xi v}.$$

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Conclusion: $\Omega = \mathbb{R}$, $d_\xi = 1$, $\mathcal{V}$ is commutative.

Spectral functions of angular Toeplitz operators

$$q_\xi(v) = \sqrt{\frac{2\xi}{1 - e^{-2\pi\xi}}}\; e^{-\xi v}.$$

$$\gamma_\sigma(\xi) = \frac{2\xi}{1 - e^{-2\pi\xi}} \int_0^\pi \sigma(v)\, e^{-2\xi v}\, \mathrm{d}v.$$

Vasilevski (2003).

Weighted dilations in $L^2_{\mathrm{harm}}(\Pi)$ and angular operators

For the harmonic Bergman space, we apply the same method and obtain

$$\Omega = \mathbb{R}, \qquad d_\xi = 2.$$

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We are unable to compute L for the last examples,
but we know the final answers from the papers of our colleagues
(Loaiza, Sánchez-Nungaray, Ramírez-Ortega, etc.)

Outline

The radial basis function kernel on $\mathbb{C}^n$

$$K_z(w) = \exp\left(-\alpha^2 \sum_{j=1}^{n} (w_j - \overline{z_j})^2\right) \qquad (z, w \in \mathbb{C}^n).$$

Here $\alpha > 0$ is a fixed number.

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The corresponding RKHS is $H = \{f \in A(\mathbb{C}^n) \colon \ \|f\| < +\infty\}$, where

$$\|f\| := \left(\frac{2^n \alpha^{2n}}{\pi^n} \int_{\mathbb{C}^n} |f(z)|^2 \exp\left(-4\alpha^2 \sum_{j=1}^{n} \operatorname{Im}(z_j)^2\right) \mathrm{d}\mu_{2n}(z)\right)^{1/2}.$$

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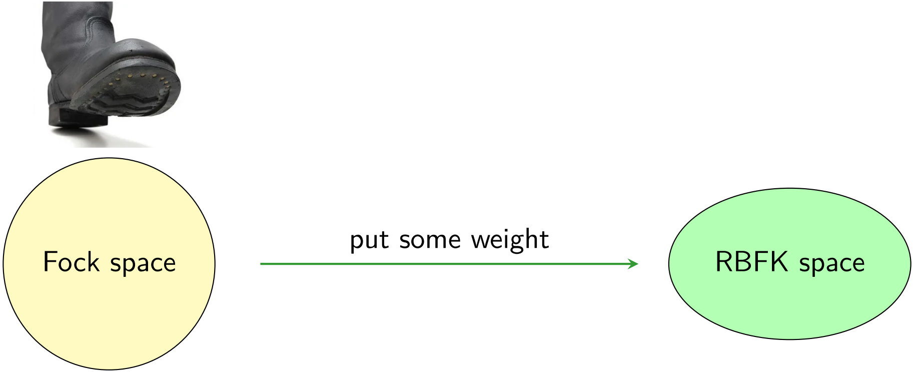
Steinwart, Hush, Scovel (2006).

RBFK space can be seen as a “flatten” Fock space



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Then the reproducing kernel takes the form

$$K_{x,y}(u, v) = \exp\left(-\alpha^2 \sum_{j=1}^{n} ((u_j - x_j)^2 - (v_j + y_j)^2 + 2\mathrm{i}(u_j - x_j)(v_j + y_j))\right).$$

Since $K_{x,y}(u, v) = K_{0,y}(u - x, v)$, the space is invariant under horizontal translations.

Vectical operators in the RBFK space

A computation with the Gaussian integral yields

$$L_{\xi,y}(v) = \left(\frac{\sqrt{\pi}}{\alpha}\right)^n \exp\left(-\sum_{j=1}^{n}\left(2\pi(v_j+y_j)\xi_j + \frac{\pi^2\xi_j^2}{\alpha^2}\right)\right).$$

The function $L_{\xi,y}(v)$ factorizes as $\overline{q_\xi(y)}q_\xi(v)$, where

$$q_\xi(v) = \left(\frac{\sqrt{\pi}}{\alpha}\right)^{n/2} \exp\left(-\sum_{j=1}^{n}\left(2\pi v_j\xi_j + \frac{\pi^2\xi_j^2}{2\alpha^2}\right)\right).$$

In this example, $\Omega = \mathbb{R}^n$, $d_\xi = 1$, and $\mathcal{V}$ is commutative.