Diagonalization of translation-invariant operators in reproducing kernel Hilbert spaces

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(thanks to Nikolai Vasilevski and Christian R. Leal Pacheco)

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Various C*-algebras generated by Toeplitz operators with symbols invariant under some group actions

- Radial operators in $\mathcal{A}^2(\mathbb{D})$.
- Vasilevski: vertical and angular operators in $\mathcal{A}^2(\Pi)$.
- Grudsky, Karapetyants, Quiroga-Barranco, Vasilevski: commutative C*-algebras of operators in $\mathcal{A}^2_\alpha(\mathbb{D})$.
- Hutník: vertical operators in wavelet spaces.
- Loaiza, Lozano: 3 classes in harmonic Bergman spaces.
- Sánchez-Nungaray, Ramírez Ortega: vertical operators in polyanalytic spaces.
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Objective

Decomposition of
W*-algebras

Translation-invariant operators
in various RKHS

Principal assumptions and results

Assumptions:

- domain $G \times Y$, where
- G is an locally compact abelian group,
- Y is a measure space,
- H is a reproducing kernel Hilbert space over $G \times Y$,
- $\mathcal{V} :=$ W*-algebra of all translation-invariant operators in H (instead of the C*-algebra of Toeplitz operators).

Results:

- decomposition of $\mathcal{V}$ into direct integral,
- commutativity criterion for $\mathcal{V}$ (in elementary terms),
- diagonalization in the commutative case.

Multiplication operators in $L^2(X,\mu)$

Given a σ-finite measure space (X,μ)
and a function $\alpha \in L^\infty(X,\mu)$,
let $M_\alpha \in \mathcal{B}(L^2(X,\mu))$ be the multiplication operator by α:

$$M_\alpha f := \alpha f.$$

Properties of M_α:

$$\|M_\alpha\| = \|\alpha\|_\infty, \qquad M_\alpha M_\beta = M_{\alpha\beta} = M_\beta M_\alpha.$$

Algebra of multiplication operators $\cong L^\infty(X,\mu)$.

Locally compact abelian groups

Let G be a LCAG, $\widehat{G} \cong$ dual group of G,
$\nu :=$ Haar measure on G, $\widehat{\nu} :=$ Haar measure on $\widehat{G}$.
Given ξ in $\widehat{G}$, denote by $E(\cdot, \xi)$ the corresponding character.
Examples:

- $G = \mathbb{R}, \quad \widehat{G} = \mathbb{R}, \quad E(a, \xi) = \mathrm{e}^{\mathrm{i}\, a\xi}, \quad \nu = \widehat{\nu} = \frac{1}{\sqrt{2\pi}} \mu_{\mathbb{R}}.$

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- $G = \mathbb{R}/(2\pi\mathbb{Z}) \simeq \mathbb{T}, \quad \widehat{G} = \mathbb{Z}, \quad E(\theta + 2\pi\mathbb{Z}, k) = \mathrm{e}^{\mathrm{i}\, k\theta}.$

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Fourier transform:

$$(Ff)(\xi) := \int_G f(x) \overline{E(x,t)}\, \mathrm{d}\nu(x) \qquad (\xi \in \widehat{G}),$$

Plancherel identity:

$$\|Ff\|_{L^2(\widehat{G}, \widehat{\nu})} = \|f\|_{L^2(G,\nu)}.$$

Operators in $L^2(G)$ invariant under translations

Given a in G, define $\tau_a \colon L^2(G) \to L^2(G)$,

$$(\tau_a f)(x) \coloneqq f(x - a).$$

Proposition (desciption of multipliers in $L^2(G)$)

Let $S \in \mathcal{B}(L^2(G))$. Then the following conditions are equiv.:

- $\forall a \in G \qquad S\tau_a = \tau_a S,$
- $\exists \sigma \in L^\infty(\widehat{G}) \qquad F^* S F = M_\sigma.$

Larsen (1971): An introduction to the theory of multipliers.

As a consequence, the W*-algebra of translation-invariant operators in $L^2(G)$ is isometrically isomorphic to $L^\infty(\widehat{G})$.

Horizontal translations in $L^2(G \times Y)$

Suppose that G is LCAG (σ-compact and metrizable),
(Y, λ) is a σ-finite measure space,
$L^2(G)$ and $L^2(Y)$ are separable.

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For each a in G define $U_a \in \mathcal{B}(L^2(G \times Y))$,

$$U_a := \tau_a \otimes I, \qquad (U_a f)(x, y) := f(x - a, y).$$

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Operators in $L^2(G \times Y)$ invariant under horizontal translations:

$$\mathcal{U} := \{S \in \mathcal{B}(L^2(G \times Y)) \colon \quad \forall a \in G \quad U_a S = S U_a\}.$$

In other words, $\mathcal{U} := \{U_a \colon a \in G\}'$.

Principal technique

Fourier transform with respect to the first coordinate:

$$\Phi := F \otimes I, \qquad (\Phi f)(\xi, y) := \int_G f(x,y)\overline{E(x,\xi)}\,\mathrm{d}\nu(x).$$

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Given S in $\mathcal{U}$, $\Phi\, S\, \Phi^*$ decomposes into a direct integral:

$$\Phi\, S\, \Phi^* = \int_{\widehat{G}}^{\oplus} A_\xi\, \mathrm{d}\widehat{\nu}(\xi), \qquad \text{with} \quad A_\xi \in \mathcal{B}(L^2(Y)).$$

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Idea of construction of A_ξ:

$$(A_\xi h)(y) := \frac{(\Phi\, S)(f \otimes h)(\xi, y)}{(F f)(\xi)}.$$

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$$(A_\xi h)(y) := \frac{(\Phi\, S)(f \otimes h)(\xi, y)}{(F f)(\xi)}.$$

More precisely, we use orthonormal bases in $L^2(G)$ and $L^2(Y)$.

Decomposition of $\mathcal{U}$

Theorem 0

$$\Phi\,\mathcal{U}\,\Phi^* = \int_{\widehat{G}}^{\oplus} \mathcal{B}(L^2(Y))\,\mathrm{d}\widehat{\nu}(\xi).$$

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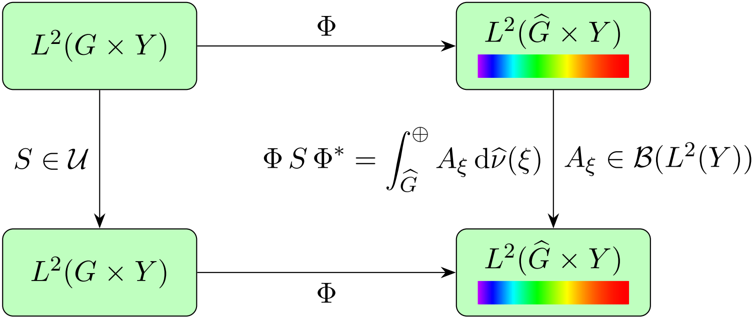
Thanks to Matthew Dawson for a short proof of this result.



RKHS over $G \times Y$, invariance under translations

Let H be a RKHS over $G \times Y$, embedded into $L^2(G \times Y)$.
$P :=$ the orthogonal projection onto H:

$$(Pf)(x,y) = \langle f, K_{x,y} \rangle = \int_G \int_Y f(x,y) \overline{K_{x,y}(u,v)} \, \mathrm{d}\nu(u) \, \mathrm{d}\lambda(v).$$

Proposition

The following conditions are equivalent:

- $\forall a \in G \qquad U_a(H) \subseteq H$;
- $\forall a \in G \qquad U_a P = P U_a, \quad$ *i.e.* $\quad P \in \mathcal{U}$;
- $\forall a, x \in G \quad \forall y \in Y \qquad U_a K_{x,y} = K_{x+a,y}$;
- $\forall x, u \in G \quad \forall y, v \in Y \qquad K_{x,y}(u,v) = K_{0,y}(u-x,v)$.

In what follows, we suppose that these conditions are fulfilled.

Translation-invariant operators in H

$V_a :=$ the translation operator $U_a = \tau_a \otimes I$ compressed to H, i.e.

$$V_a \in \mathcal{B}(H), \qquad (V_a f)(x,y) := f(x-a,y).$$

The family $(V_a)_{a\in G}$ is a unitary representation of G in H.

$\mathcal{V} :=$ the W*-algebra of operators in H that commute with these translations:

$$\mathcal{V} := \{S \in \mathcal{B}(H) \colon \quad \forall a \in G \quad V_a S = S V_a\}.$$

In other words, $\mathcal{V} = \{V_a \colon a \in G\}'$.

Example: Bergman space over $\mathbb{R} \times \mathbb{R}_+$

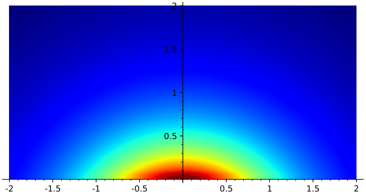
$$G := \mathbb{R}, \quad \widehat{G} := \mathbb{R}, \quad E(x,\xi) := \mathrm{e}^{\mathrm{i}\,x\xi}, \quad Y := \mathbb{R}_+,$$

$$\nu = \widehat{\nu} = \frac{\mu_{\mathbb{R}}}{\sqrt{2\pi}}, \qquad \lambda := \sqrt{2\pi}\,\mu_{\mathbb{R}_+}, \qquad \nu \otimes \lambda = \mu_{\mathbb{R}\times\mathbb{R}_+}.$$

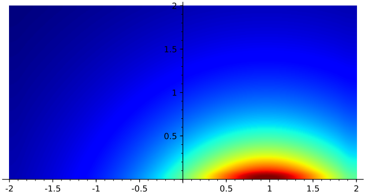
$$H = \mathcal{A}^2(\mathbb{R} \times \mathbb{R}_+) := \text{Bergman space over } \mathbb{R} \times \mathbb{R}_+,$$

$$K_{(x,y)}(u,v) = -\frac{1}{\pi(u - x + \mathrm{i}(v+y))^2}.$$

Example: Bergman kernel



$|K_{0,1}|$

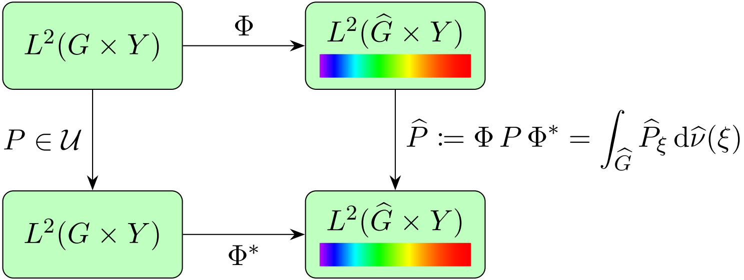


$|K_{1,1}|$

We see that $K_{x+a,y} = U_a K_{x,y}$.

In this example, the operators V_a are horizontal translations compressed to H, and $\mathcal{V} =$ vertical operators in H.

Decomposition of $\widehat{H}$ into fibers



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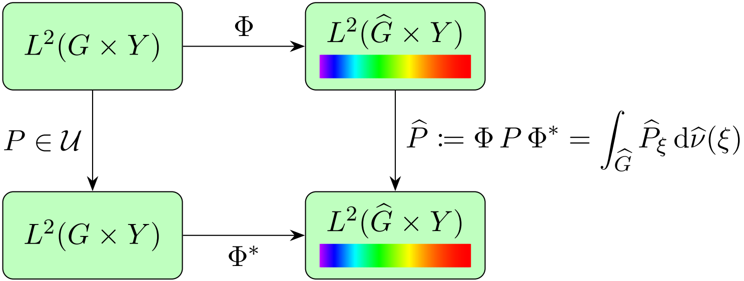
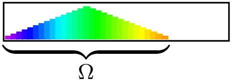


Denote by $\widehat{H}_\xi$ the image of the projection $\widehat{P}_\xi$
and by Ω the set of the frequences ξ such that $\widehat{H}_\xi$ is not trivial:

$$\widehat{H}_\xi := \widehat{P}_\xi(L^2(Y)), \qquad \Omega := \{\xi \in \widehat{G} \colon \ \dim(\widehat{H}_\xi) > 0\}.$$

Then

$$\widehat{H} = \int_\Omega^\oplus \widehat{H}_\xi \,\mathrm{d}\widehat{\nu}(\xi).$$



Fourier image of the reproducing kernel

$$\overline{K_{(x,y)}(u,v)} = K_{(u,v)}(x,y) = K_{(0,v)}(x-u,y).$$

$$(Pf)(x,y) = \int_Y \int_G f(u,v) K_{(0,v)}(x-u,y)\, \mathrm{d}\nu(u) \mathrm{d}\lambda(v).$$

We have a convolution over the first coordinate.
Therefore we apply $\Phi = F \otimes I$.

Put $L_{\xi,y}(v) \coloneqq (\Phi K_{(0,y)})(\xi, v)$, i.e.,

$$L_{\xi,y}(v) \coloneqq \int_G \overline{E(u,\xi)} K_{(0,y)}(u,v)\, \mathrm{d}\nu(u).$$

Each fiber is a RKHS

Proposition

Under some technical conditions over K*,*
$L_{\xi,y}(v)$ *depends continuously on* ξ*,*
$(L_{\xi,y})_{y\in Y}$ *is a reproducing kernel of* $\widehat{H}_\xi$*:*

$$(\widehat{P}_\xi h)(y) = \langle h, L_{\xi,y}\rangle = \int_Y h(v)\overline{L_{\xi,y}(v)}\,\mathrm{d}\lambda(v).$$

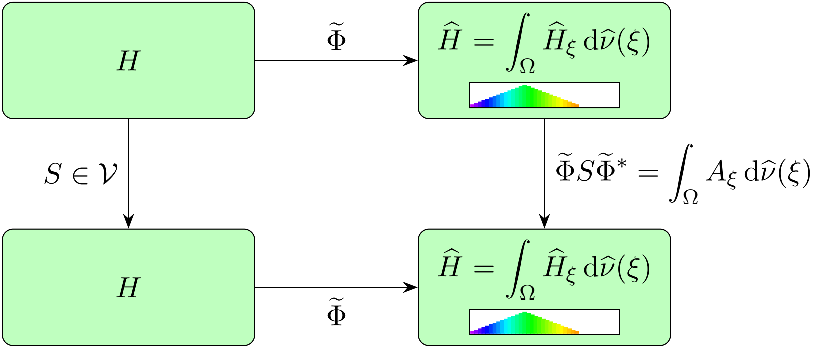
As a consequence,

$$\dim(\widehat{H}_\xi) = \int_Y L_{\xi,v}(v)\,\mathrm{d}\lambda(v).$$

Decomposition of the algebra $\mathcal{V}$

Theorem 1

$$\widetilde{\Phi}\mathcal{V}\widetilde{\Phi}^* = \int_\Omega^\oplus \mathcal{B}(\widehat{H}_\xi)\,\mathrm{d}\widehat{\nu}(\xi).$$



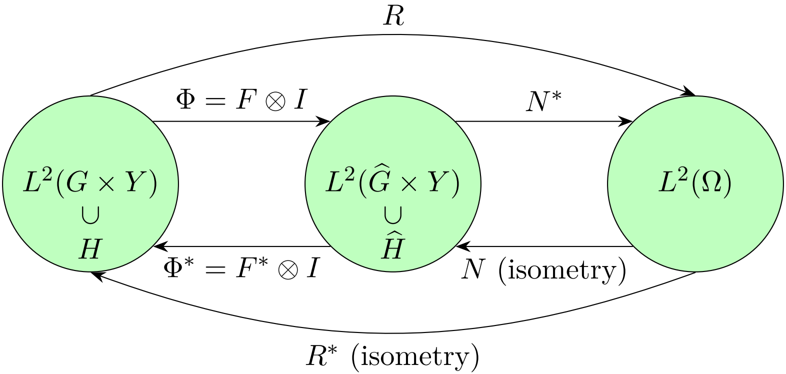
Commutativity criterion for $\mathcal{V}$

Theorem 2

The following conditions are equivalent:

- $\mathcal{V}$ *is commutative;*
- $\forall \xi \in \Omega \qquad \dim(\widehat{H}_\xi) = 1;$
- $\forall \xi \in \Omega$
$$\int_Y L_{\xi,v}(v)\,\mathrm{d}\lambda(v) = 1;$$
- $\forall \xi \in \Omega \quad \forall y, v \in Y$
$$|L_{\xi,y}(v)|^2 = L_{\xi,y}(y) L_{\xi,v}(v);$$
- *there exists a measurable function* $q \colon \Omega \times Y \to \mathbb{C}$ *such that*
$$L_{\xi,y}(v) = \overline{q_\xi(y)} q_\xi(v).$$

Commutative case (one-dimensional fibers)



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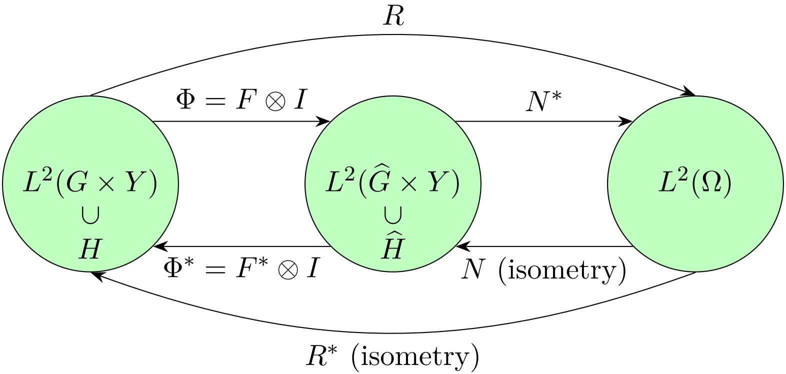
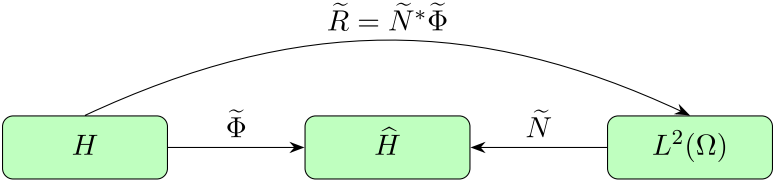


Here

$$(Nh)(\xi, y) := \begin{cases} q_\xi(y)h(\xi), & \xi \in \Omega; \\ 0, & \xi \in \widehat{G} \setminus \Omega. \end{cases}$$

Commutative case: $H \cong \widehat{H} \cong L^2(\Omega)$

After restricting the domains we obtain unitary operators:



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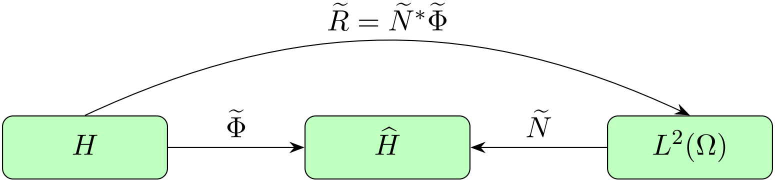
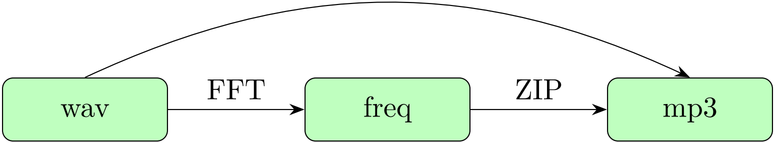


Analogy with transforming the sound in the mp3 or ogg format:

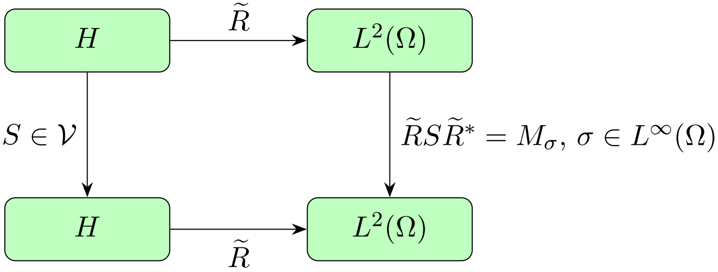


Diagonalization of $\mathcal{V}$ in the commutative case

Theorem 3

Suppose that $\dim(\widehat{H}_\xi) = 1$ *for every* ξ *in* Ω. *Then*

$$\mathcal{V} \cong L^\infty(\Omega).$$

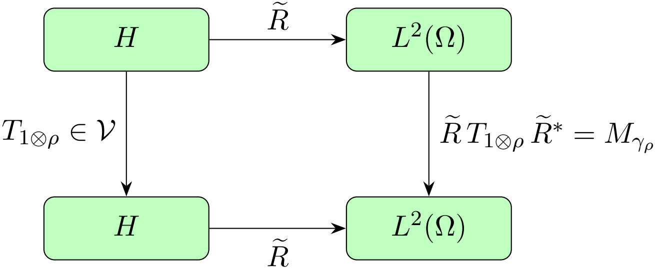


Toeplitz operators with invariant symbols

Corollary: diagonalization of Toeplitz operators

Let $\rho \in L^\infty(Y)$. *Then* $T_{1\otimes\rho} \in \mathcal{V}$ *and* $\widetilde{R}\, T_{1\otimes\rho}\, \widetilde{R}^* = M_{\gamma_\rho}$, *where*

$$\gamma_\rho(\xi) = \int_Y \rho(v) \underbrace{L_{\xi,v}(v)}_{|q_\xi(v)|^2} \, \mathrm{d}\lambda(v).$$



Example: vertical operators in $\mathcal{A}^2(\mathbb{R} \times \mathbb{R}_+)$

$$L_{\xi,y}(v) = \sqrt{\frac{2}{\pi}}\,\xi\;\mathrm{e}^{-\xi(y+v)}\;1_{\mathbb{R}_+}(\xi) = \overline{q_\xi(y)}\,q_\xi(v),$$

where

$$q_\xi(v) = \left(\frac{2}{\pi}\right)^{1/4}\sqrt{\xi}\;\mathrm{e}^{-\xi v} \qquad (\xi > 0).$$

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So, $\Omega = \mathbb{R}_+$, $\mathcal{V}$ is commutative, $\mathcal{V} \cong L^\infty(\mathbb{R}_+)$.

$$\gamma_\rho(\xi) = \int_{\mathbb{R}_+} \rho(v) L_{\xi,v}(v)\,\mathrm{d}\lambda(v) = \int_{\mathbb{R}_+} \rho(v)\ 2\xi\ \mathrm{e}^{-2\xi v}\ \mathrm{d}v.$$

Thereby we reproduce a result found by Vasilevski (1999).

Example: vertical Toeplitz operators in $\mathcal{A}^2(\mathbb{R} \times \mathbb{R}_+)$

Now consider $\mathcal{VT}$:= the C*-algebra generated by Toeplitz operators with bounded generated symbols. Thanks to the diagonalization explained above,

$$\mathcal{VT} \cong \text{C}^*\text{-alg}(\Gamma), \qquad \Gamma := \{\gamma_\rho \colon \ \rho \in L^\infty(\mathbb{R})\}.$$

Herrera Yañez, M, Vasilevski (2013) proved Γ is dense in the C*-algebra $\text{LogOsc}(\mathbb{R}_+)$ of all bounded log-oscillating functions on $\mathbb{R}_+$. So,

$$\mathcal{V} \cong L^\infty(\mathbb{R}_+), \qquad \mathcal{VT} \cong \text{LogOsc}(\mathbb{R}_+).$$

Notice that $\mathcal{VT}$ is weakly dense in $\mathcal{V}$. A more general result was already proved by Dawson, Ólafsson and Quiroga-Barranco, using the result by Engliš.

Example: vertical operators in $h^2(\mathbb{R} \times \mathbb{R}_+)$

For the vertical operators in the harmonic Bergman space,

$$L_{\xi,y}(v) = \sqrt{\frac{2}{\pi}}\,|\xi|\,\mathrm{e}^{-|\xi|(y+v)} = \overline{q_\xi(y)}\,q_\xi(v),$$

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So, $\Omega=\mathbb{R}$, $\mathcal{V}$ is commutative, $\mathcal{V}\cong L^\infty(\mathbb{R})$.

$$\gamma_\rho(\xi) = \int_{\mathbb{R}_+} \rho(v)\;2|\xi|\;\mathrm{e}^{-2|\xi|v}\,\mathrm{d}v.$$

Thereby we reproduce a result by Loaiza and Lozano (2013).

Example: vertical Toeplitz operators in $h^2(\mathbb{R} \times \mathbb{R}_+)$

For the harmonic Bergman space, the functions γ_ρ are even. Thus,

$$\mathcal{VT} \cong \text{EvenLogOsc}(\mathbb{R}),$$

where $\text{EvenLogOsc}(\mathbb{R})$ consists of even functions σ such that $\sigma|_{\mathbb{R}_+} \in \text{LogOsc}(\mathbb{R}_+)$.

Therefore $\mathcal{VT}$ is not weakly dense in $\mathcal{V}$.