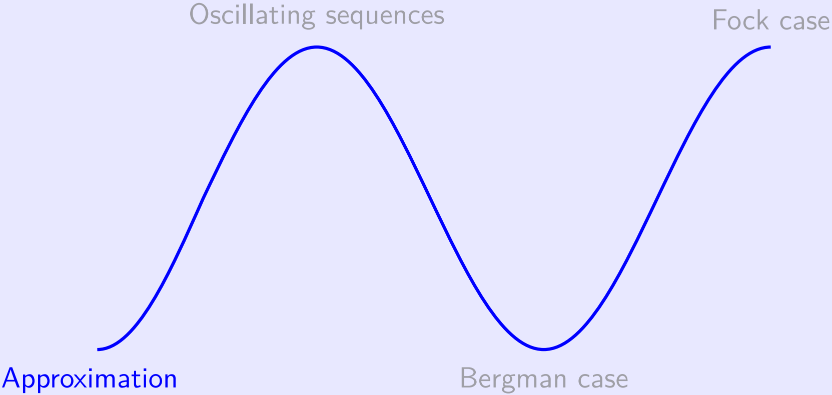
Eigenvalues of radial Toeplitz operators, log-oscillation and sqrt-oscillation

Egor Maximenko

based on joint works with Kevin Esmeral, Sergei Grudsky, Crispin Herrera Yañez and Nikolai Vasilevski

Instituto Politécnico Nacional, ESFM, México

Querétaro, México
WiS&E 2016


Oscillating sequences
Fock case
Approximation
Bergman case

Bounded uniformly continuous functions

Modulus of continuity

Given $f\colon \mathbb{R} \to \mathbb{C}$, define $\omega_f\colon [0,+\infty) \to [0,+\infty)$ by

$$\omega_f(\delta) \coloneqq \sup\{|f(x) - f(y)|\colon \quad x, y \in \mathbb{R}, \quad |x - y| \le \delta\}.$$

Bounded uniformly continuous functions

$$\mathrm{C}_u(\mathbb{R}) \coloneqq \Big\{ f \in \mathbb{C}^{\mathbb{R}}\colon \quad \sup_{x\in\mathbb{R}} |f(x)| < +\infty \quad \wedge \quad \lim_{\delta\to 0^+} \omega_f(\delta) = 0 \Big\}.$$

Proposition. $\mathrm{C}_u(\mathbb{R})$ is a C^*-subalgebra of $L^\infty(\mathbb{R})$.

Convolution, Fourier transform

Convolution

Given $k \in L^1(\mathbb{R})$ and $f \in L^\infty(\mathbb{R})$,

$$(k * f)(x) := \int_{\mathbb{R}} k(x-y) f(y) \, \mathrm{d}y.$$

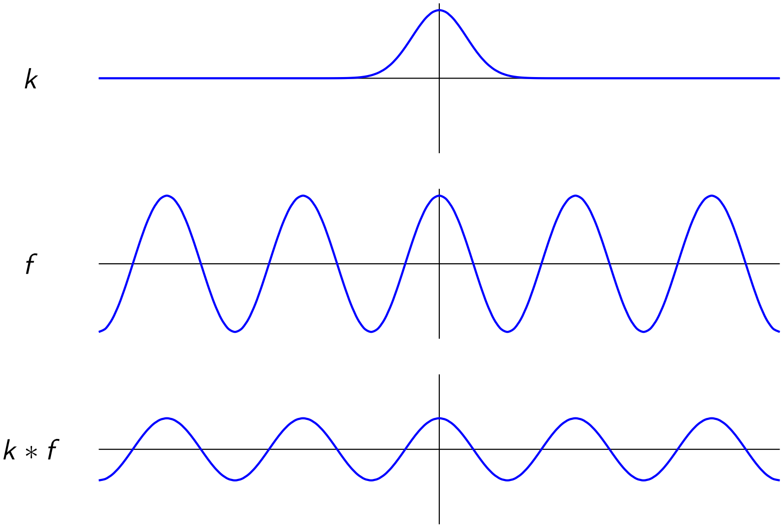
It is easy to see that $k * f \in \mathsf{C}_u(\mathbb{R})$.

Fourier transform

Given $k \in L^1(\mathbb{R})$,

$$\widehat{k}(t) := \int_{\mathbb{R}} k(x) \, \mathrm{e}^{-2\pi \, \mathrm{i} \, xt} \, \mathrm{d}x.$$

Convolution and oscillations: example



Bounded uniformly continuous functions can be uniformly approximated by convolutions

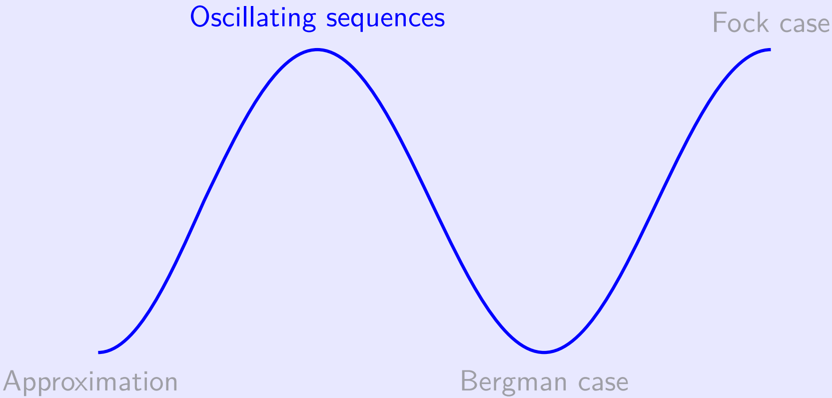
Approximation Theorem. Let $k \in L^1(\mathbb{R})$ satisfy Wiener's condition:

$$\forall t \in \mathbb{R} \qquad \widehat{k}(t) \neq 0.$$

Then $\{k * f: \ f \in L^\infty(\mathbb{R})\}$ is a dense subset of $\mathrm{C}_u(\mathbb{R})$.

We have not found this result in the literature, so we included a proof in Esmeral, Maximenko (2016) doi:10.1007/s11785-016-0557-0

The theorem is very close to Tauberian Wiener's theorem,
and our proof is based on the well-known
"Wiener's approximate deconvolution technique":
Dirac sequences and Wiener's Division Lemma.


Oscillating sequences
Fock case
Approximation
Bergman case

Log-oscillating sequences (Robert Schmidt, 1924)

Definition. LO $:=$ bounded sequences $x = (x_n)_{n=0}^{\infty}$ such that

$$\lim_{\frac{m+1}{n+1}\to 1} |x_m - x_n| = 0.$$

In other words, LO is the set of all bounded functions $\mathbb{N}_0 \to \mathbb{C}$ that are uniformly continuous with respect to the log-metric:

$$\rho_{\log}(m,n) := |\log(m+1) - \log(n+1)|,$$

$$\omega_{\rho_{\log},x}(\delta) := \sup\{|x_m - x_n| : \ m,n \in \mathbb{N}_0,\ \rho_{\log}(m,n) \le \delta\},$$

$$\mathrm{LO} := \{x \in \ell^\infty : \ \lim_{\delta\to 0^+} \omega_{\rho_{\log},x}(\delta) = 0\}.$$

Proposition. LO is a C*-subalgebra of ℓ^∞.

Log-oscillating sequences can be obtained from bounded uniformly continuous functions

Proposition. The operator $\Phi\colon\ C_u(\mathbb{R}) \to \mathrm{LO}$, defined by

$$(\Phi f)_n := f(\log(n+1)),$$

is an epimorphism of C^*-algebras.

Idea of proof of the surjective property:
Given a sequence $x \in \mathrm{LO}$, put

$$f(\log(n+1)) := x_n \qquad (n \in \mathbb{N}_0)$$

and apply the linear interpolation between the nodes $\log(n+1)$.

Sqrt-oscillating sequences

RO $:=$ the set of all bounded sequences that are uniformly continuous with respect to the sqrt-metric:

$$\rho_{\mathrm{sqrt}}(m,n) := |\sqrt{m} - \sqrt{n}|.$$

Proposition. RO is a C*-subalgebra of ℓ^∞.

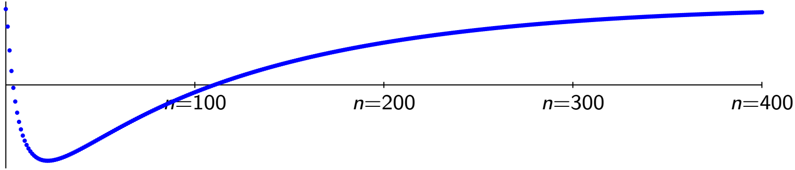
Proposition. The operator $\Psi\colon\ \mathrm{C}_u(\mathbb{R}) \to \mathrm{RO}$, defined by

$$(\Psi f)_n := f(\sqrt{n}),$$

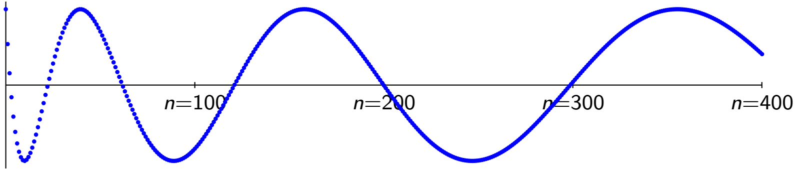
is an epimorphism of C*-algebras.

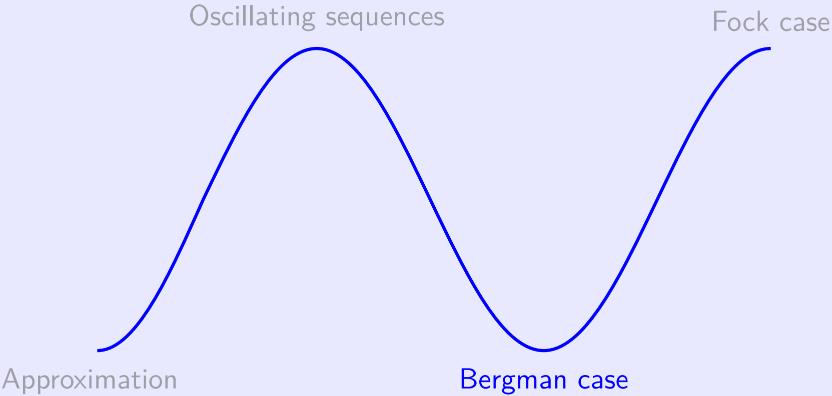
Example of log-oscillation and sqrt-oscillation

$x_n = \cos(\log(n+1))$:



$x_n = \cos(\sqrt{n})$:




Oscillating sequences
Fock case
Approximation
Bergman case

Bergman space on the unit disk

$\mathbb{D} := \{z \in \mathbb{C} \colon \ |z| < 1\}$.

$\mu_{\mathbb{D}} := \dfrac{1}{\pi}$ Lebesgue plane measure.

$\mathcal{A}^2(\mathbb{D}) := \{f \in H(\mathbb{D}) \colon \ f \in L^2(\mathbb{D})\}$.

Bergman projection $P_{\mathbb{D}}$:= orthogonal projection $L^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D})$.

Orthonormal basis in $\mathcal{A}^2(\mathbb{D})$:

$$e_n(z) = \sqrt{n+1}\, z^n.$$

Given $g \in L^\infty(\mathbb{D})$, denote by T_g the Toeplitz operator

$$T_g \colon \mathcal{A}^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D}), \qquad T_g f := P_{\mathbb{D}}(gf).$$

Radial Toeplitz operators on the Bergman space

Given $a \in L^\infty([0,1))$, denote by $\widetilde{a}$ its extension onto $\mathbb{D}$:

$$g(z) := \widetilde{a}(z) := a(|z|) \qquad (z \in \mathbb{D}).$$

Diagonalization of radial Toeplitz operators on the Bergman space:

$$T_g e_n = \lambda_a(n) e_n,$$

where

$$\lambda_a(n) = (n+1) \int_0^1 a(\sqrt{r})\, r^n \,\mathrm{d}r.$$

Korenblum, Zhu (1995)

http://www.mathjournals.org/jot/1995-033-002/1995-033-002-010.html

From radial Toeplitz operators to eigenvalues' sequences

$$T_{\widetilde{a}} \quad \longleftrightarrow \quad \lambda_a$$

$$\{T_{\widetilde{a}}:\ a \in L^\infty([0,1))\} \quad \longleftrightarrow \quad \Lambda := \{\lambda_a:\ a \in L^\infty([0,1))\}$$

$$\mathrm{C}^*\text{-alg}\{T_{\widetilde{a}}:\ a \in L^\infty(0,1)\} \quad \longleftrightarrow \quad \mathrm{C}^*\text{-alg}(\Lambda)$$

Natural questions:

What is the closure of Λ ?

What is the C^*-algebra generated by Λ ?

From radial Toeplitz operators to eigenvalues' sequences

$$T_{\widetilde{a}} \longleftrightarrow \lambda_a$$

$$\{T_{\widetilde{a}}\colon\ a \in L^\infty([0,1))\} \longleftrightarrow \Lambda := \{\lambda_a\colon\ a \in L^\infty([0,1))\}$$

$$\text{C}^*\text{-alg}\{T_{\widetilde{a}}\colon\ a \in L^\infty(0,1)\} \longleftrightarrow \text{C}^*\text{-alg}(\Lambda)$$

Natural questions:

What is the closure of Λ ?

What is the C*-algebra generated by Λ ?

These two questions have the same answer: LO.

Key idea: spectral sequences as convolutions

$$\lambda_a(n) = (n+1)\int_0^1 a(\sqrt{r})\, r^n \,\mathrm{d}r.$$

Change of variables:

$$n+1 = \exp(x), \qquad r = \exp(-\exp(-y)).$$

$$\lambda_a(n) = \int_{\mathbb{R}} \underbrace{a(\exp(-\exp(-y)/2)}_{b(y)} \, k(x-y)\,\mathrm{d}y = (k*b)(x),$$

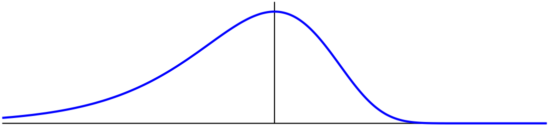
where

$$k(x) = \frac{\mathrm{e}^x}{\mathrm{e}^{\mathrm{e}^x}}.$$

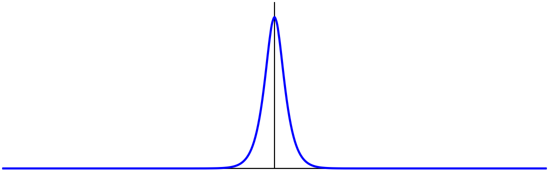
The convolution kernel satisfies Wiener's condition

$$k(x) = \frac{e^x}{e^{e^x}}, \qquad \widehat{k}(t) = \Gamma(1 - 2\,i\,\pi t) \neq 0 \qquad (t \in \mathbb{R}).$$

k



$|\widehat{k}|$



Main result for the radial Bergman case

Theorem. $\underbrace{\{\lambda_a : \ a \in L^\infty([0,1))\}}_{\Lambda}$ is a dense subset of LO.

As a consequence, $C^*(\Lambda) = \text{LO}$, and the C*-algebra generated by radial Toeplitz operators on $\mathcal{A}^2(\mathbb{D})$ is isometrically isomorphic to LO.

Suárez (2008) doi:10.1112/blms/bdn042

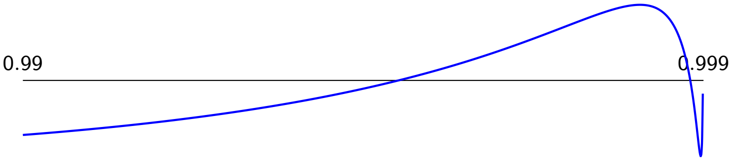
Grudsky, M, Vasilevski (2013) euclid.cma/1356039033

Bauer, Herrera Yañez, Vasilevski (2014) doi:10.1007/s00020-013-2101-1

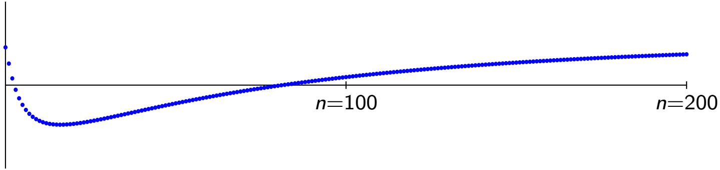
Herrera Yañez, M, Vasilevski (2015) doi:10.1007/s00020-014-2213-2

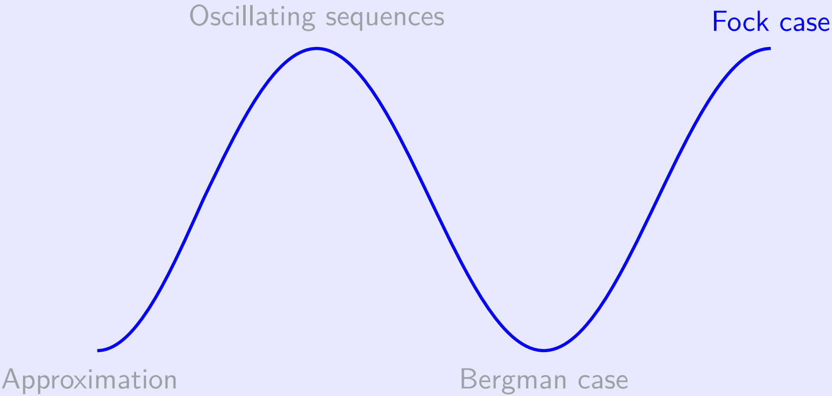
Example of spectral sequence in radial Bergman case

The function $a(r) = \cos(\log(-2\log(r))$ oscillates rapidly near $r = 1$.



Then $\lambda_a(n) = \mathrm{Re}(\Gamma(1 - \mathrm{i})\exp(\mathrm{i}\log(n + 1)))$.




Oscillating sequences
Fock case
Approximation
Bergman case

Bargmann–Segal–Fock space

Gaussian measure $\mathrm{d}\gamma(z) = \frac{1}{\pi}\ \mathrm{e}^{-|z|^2}\ \mathrm{d}\mu(z)$.

$\mathcal{A}^2(\mathbb{C}, \mathrm{d}\gamma) := \{f \in H(\mathbb{C}):\ f \in L^2(\mathbb{C}, \mathrm{d}\gamma)\}$.

$P_{\mathbb{C}} :=$ orthogonal projection $L^2(\mathbb{C}, \mathrm{d}\gamma) \to \mathcal{A}^2(\mathbb{C}, \mathrm{d}\gamma)$.

Orthonormal basis in $\mathcal{A}^2(\mathbb{C}, \mathrm{d}\gamma)$:

$$f_n(z) = \frac{z^n}{\sqrt{n!}}.$$

Given $g \in L^\infty(\mathbb{C})$, denote by T_g the Toeplitz operator

$$T_g\colon \mathcal{A}^2(\mathbb{C}, \mathrm{d}\gamma) \to \mathcal{A}^2(\mathbb{C}, \mathrm{d}\gamma), \qquad T_g f := P_{\mathbb{C}}(gf).$$

Radial Toeplitz operators on the Fock space

Given $a \in L^\infty(\mathbb{R}_+)$, denote by $\widetilde{a}$ its extension onto $\mathbb{C}$:

$$g(z) := \widetilde{a}(z) := a(|z|) \qquad (z \in \mathbb{C}).$$

Diagonalization of radial Toeplitz operators on the Fock space:

$$T_g f_n = \xi_a(n) f_n,$$

where

$$\xi_a(n) = \frac{1}{\sqrt{n!}} \int_{\mathbb{R}_+} a(\sqrt{r})\, \mathrm{e}^{-r}\, r^n \,\mathrm{d}r.$$

Grudsky, Vasilevski (2002) doi:10.1007/BF01197858

Key idea: approximate spectral sequences by convolutions

$$\xi_a(n) - \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}_+} \mathrm{e}^{-2(\sqrt{n}-y)^2} \, a(y) \, \mathrm{d}y \to 0 \quad \text{as} \quad n \to \infty.$$

Let a_{ext} be the function a extended from $\mathbb{R}_+$ onto $\mathbb{R}$ by zero. Then

$$\xi_a(n) \approx (k * a_{\mathrm{ext}})(\sqrt{n}) \quad \text{as} \quad n \to \infty,$$

where k is the following Gaussian density:

$$k(x) = \sqrt{\frac{2}{\pi}} \, \mathrm{e}^{-2x^2} .$$

The Fourier transform of k does not vanish:

$$\widehat{k}(t) = \mathrm{e}^{-\frac{1}{2}\pi^2 t^2} .$$

Main result for the radial Fock case

Theorem. $\{\xi_a\colon\ a \in L^\infty(\mathbb{R}_+)\}$ is a dense subset of RO.

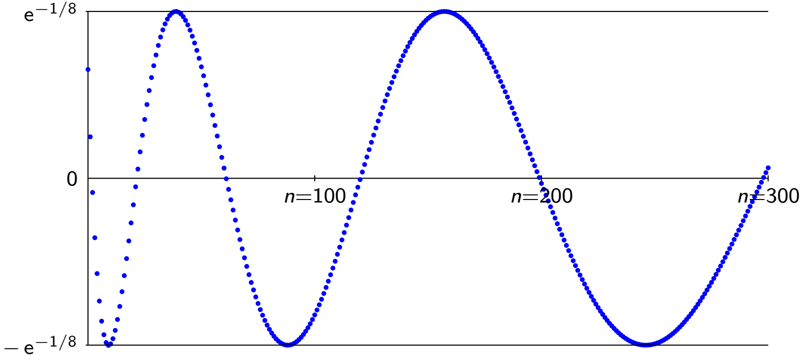
As a consequence, the C*-algebra generated by radial Toeplitz operators on $\mathcal{A}^2(\mathbb{C}, \mathrm{d}\gamma)$ is isometrically isomorphic to RO.

Esmeral, Maximenko (2016) doi:10.1007/s11785-016-0557-0

Example of spectral sequence in radial Fock case

Let $a(r) = \cos(r)$, $r \in \mathbb{R}_+$.
Then the eigenvalue sequence ξ_a has a typical $\sqrt{}$-oscillation:

$$\xi_a(n) = {}_1F_1(1+n; 1/2; 1/4) \approx \mathrm{e}^{-1/8}\,\cos(\sqrt{n}) + o(1).$$



Summary

C*-algebra generated by radial Toeplitz operators on the Bergman space ↔ log-oscillating sequences

C*-algebra generated by radial Toeplitz operators on the Fock space ↔ $\sqrt{}$-oscillating sequences