Approximate deconvolution and density of vertical Toeplitz operators

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Outline

Dirac sequences
and approximate deconvolution

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Density of Λ_f in $\mathrm{UC}(\mathbb{R})$
where $\Lambda_f := \{f * a\colon\ a \in L^\infty(\mathbb{R})\}$

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Density of Toeplitz operators
(parabolic case)

Dirac sequences

Definition

A *Dirac sequence* is a sequence $(u_n)_{n\in\mathbb{N}}$ in $L^1(\mathbb{R})$ such that:

- $u_n(x) \geq 0$ for every $x \in \mathbb{R}$, $n \in \mathbb{N}$.
- $\displaystyle\int_{\mathbb{R}} u_n(x)\,dx = 1$ for every $n \in \mathbb{N}$.
- For every $\delta > 0$, $\displaystyle\lim_{n\to\infty} \int_{|x|\geq\delta} u_n(x)\,dx = 0$.

Some properties of Dirac sequences:

1. If $f \in L^1(\mathbb{R})$, then $\|f * u_n - f\|_1 \to 0$.
2. If $1 < p < +\infty$ and $f \in L^p(\mathbb{R})$, then $\|f * u_n - f\|_p \to 0$.
3. If $f \in \mathrm{UC}(\mathbb{R})$, then $\|f * u_n - f\|_\infty \to 0$.

Theorem 1 (approximate deconvolution for the Schwartz class)

$f \in \mathcal{S}(\mathbb{R})$, $0 \notin \widehat{f}(\mathbb{R})$ $\Longrightarrow$ $\exists$ *sequence* $(g_n)_{n\in\mathbb{N}}$ *in* $\mathcal{S}(\mathbb{R})$ *such that* $(f * g_n)_{n\in\mathbb{N}}$ *is a Dirac sequence*

Proof.

Let $(u_n)_{n\in\mathbb{N}}$ be a Dirac sequence such that $\operatorname{supp}(\widehat{u}_n)$ are compacts. Put

$$G_n := \frac{\widehat{u}_n}{\widehat{f}}.$$

$G_n \in C^\infty(\mathbb{R})$ and $\operatorname{supp}(G_n)$ is compact, therefore $G_n \in \mathcal{S}(\mathbb{R})$.
Since the class $\mathcal{S}(\mathbb{R})$ is Fourier-invariant, $\exists g_n \in \mathcal{S}(\mathbb{R})$ such that $\widehat{g}_n = G_n$.

$$\widehat{f * g_n} = \widehat{f}\, G_n = \widehat{u}_n \qquad \Longrightarrow \qquad f * g_n = u_n. \qquad \square$$

Note: Theorem 1 stays true for $f \in L^1(\mathbb{R})$, $0 \notin \widehat{f}(\mathbb{R})$.

T2 (convolutions are dense in $\mathrm{UC}(\mathbb{R})$)

$$\left.\begin{array}{l} f \in \mathcal{S}(\mathbb{R}) \\ 0 \notin \widehat{f}(\mathbb{R}) \end{array}\right. \quad \Longrightarrow \quad \Lambda_f \quad \text{is a dense subset of } \mathrm{UC}(\mathbb{R})$$

$$\Lambda_f := \{f * a\colon\ a \in L^\infty(\mathbb{R})\}$$

Proof.

By properties of the convolution, Λ_f is a subset of $\mathrm{UC}(\mathbb{R})$.

Let $\sigma \in \mathrm{UC}(\mathbb{R})$. We have to approximate σ by elements of Λ_f. Construct $(g_n)_{n=1}^\infty$ and $(u_n)_{n=1}^\infty$ as in Theorem 1, and put

$$a_n = g_n * \sigma.$$

Then $a_n \in L^\infty(\mathbb{R})$ and

$$f * a_n = f * g_n * \sigma = u_n * \sigma \quad \xrightarrow{\text{in } \mathrm{UC}(\mathbb{R})} \sigma.$$

□

Density of Toeplitz operators (unweighted parabolic case)

Vasilevski proved that all vertical Toeplitz operators acting on $\mathcal{A}^2(\Pi)$ can be diagonalized by one unitary operator $R\colon \mathcal{A}^2(\Pi) \to L^2(\mathbb{R}_+)$.

$$RT_bR^{-1} = M_{\gamma_b}, \qquad \gamma_b(x) = \int_0^{+\infty} b\left(\frac{u}{2x}\right) \mathrm{e}^{-u}\, du.$$

Denote by $\mathfrak{G}$ the set of the functions γ_b:

$$\mathfrak{G} := \{\gamma_b\colon\ b \in L^\infty(\mathbb{R}_+)\}.$$

Natural problems:

- describe the closure of $\mathfrak{G}$ in $L^\infty(\mathbb{R}_+)$;
- describe the C^*-algebra generated by $\mathfrak{G}$.

Density of Toeplitz operators (unweighted parabolic case)

$$\gamma_b(x) = \int_0^{+\infty} b\left(\frac{u}{2x}\right) e^{-u}\, du.$$

Make logarithmic changes of variables:

$$b(x/2) = a(-\log(x)), \qquad \gamma_b(x) = \xi_a(\log(x)).$$

The function $\xi_a = \gamma_b \circ \exp$ can be written as a convolution on the real line:

$$\xi_a = f * a, \qquad \text{where} \qquad f(x) = \frac{e^x}{e^{e^x}}.$$

The Fourier transform of f does not vanish:

$$\widehat{f}(t) = \frac{\Gamma(1 - i\, t)}{\sqrt{2\pi}} \neq 0 \quad (t \in \mathbb{R}).$$

By Theorem 2, $\{\xi_a\colon\ a \in L^\infty\}$ is a dense subset of $\mathrm{UC}(\mathbb{R})$.