Avram–Parter and Szegő limit theorems turned inside out

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Toeplitz matrices
Szegő's
limit theorem
Quantile function
Uniform
convergence

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Toeplitz matrices

$$T_5(a) = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} \\ a_3 & a_2 & a_1 & a_0 & a_{-1} \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}.$$

It is convenient to think that a_j are the Fourier coefficients of a certain function a defined on $[0, 2\pi]$:

$$a_j = \frac{1}{2\pi} \int_0^{2\pi} a(\theta)\, e^{-ji\theta}\, d\theta.$$

The function a is called the *generating symbol* of the matrices

$$T_n(a) = \left[a_{j-k}\right]_{j,k=1}^{n}.$$

Hermitian Toeplitz matrices, real bounded symbols

We suppose that the generating symbol is bounded and real:

$$a \in L^{\infty}([0, 2\pi], \mathbb{R}).$$

In this case the matrices are Hermitian:

$$a_{-k} = \overline{a_k}, \qquad a_0 \in \mathbb{R}.$$

$$T_5(a) = \begin{bmatrix} a_0 & \overline{a_1} & \overline{a_2} & \overline{a_3} & \overline{a_4} \\ a_1 & a_0 & \overline{a_1} & \overline{a_2} & \overline{a_3} \\ a_2 & a_1 & a_0 & \overline{a_1} & \overline{a_2} \\ a_3 & a_2 & a_1 & a_0 & \overline{a_1} \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}.$$


stealed from
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$$\begin{bmatrix} a_0 & \overline{a_1} & \overline{a_2} & \overline{a_3} \\ a_1 & a_0 & \overline{a_1} & \overline{a_2} \\ a_2 & a_1 & a_0 & \overline{a_1} \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix}$$

To understand an Hermitian matrix is...

$$
\begin{bmatrix} a_0 & \overline{a_1} & \overline{a_2} & \overline{a_3} \\ a_1 & a_0 & \overline{a_1} & \overline{a_2} \\ a_2 & a_1 & a_0 & \overline{a_1} \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix} \sim \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}
$$

To understand an Hermittian matrix is...

...to know the behavior of its

***eigenvalues** and **eigenvectors**.*

Eigenvalues of Hermitian Toeplitz matrices

For a in $L^\infty([0,2\pi],\mathbb{R})$,

$$\mathrm{sp}\ T_n(a) \subseteq [\operatorname{ess\,inf}(a), \operatorname{ess\,sup}(a)].$$

Widom (1994):

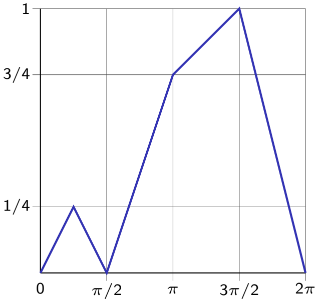
$$\mathrm{sp}\ T_n(a) \xrightarrow{\text{in the Hausdorff distance}} [\operatorname{ess\,inf}(a), \operatorname{ess\,sup}(a)].$$

$\lambda_1^{(n)}, \ldots, \lambda_n^{(n)} :=$ the eigenvalues of $T_n(a)$ written in the ascending order:

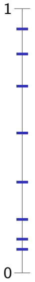
$$\lambda_1^{(n)} \leq \ldots \leq \lambda_n^{(n)}.$$

Behavior of the eigenvalues of Hermitian Toeplitz matrices

Graph of a

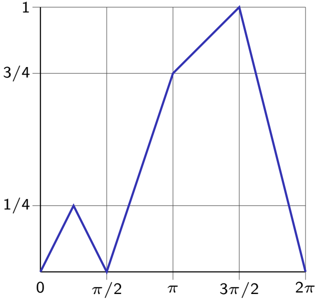


Eigenvalues of $T_8(a)$

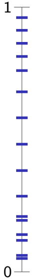


Behavior of the eigenvalues of Hermitian Toeplitz matrices

Graph of a

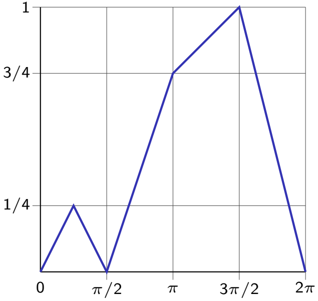


Eigenvalues of $T_{16}(a)$

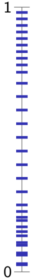


Behavior of the eigenvalues of Hermitian Toeplitz matrices

Graph of a

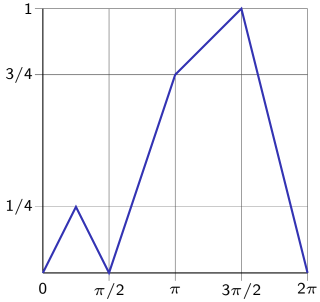


Eigenvalues of $T_{32}(a)$

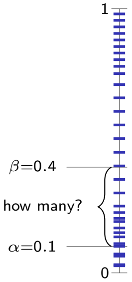


Behavior of the eigenvalues of Hermitian Toeplitz matrices

Graph of a



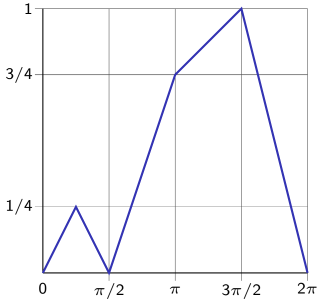
Eigenvalues of $T_{32}(a)$



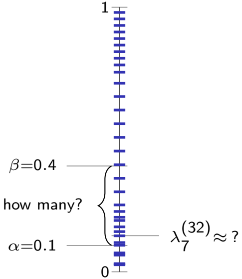
First question: How many eigenvalues are in $[\alpha, \beta]$? (Szegő, 1920).

Behavior of the eigenvalues of Hermitian Toeplitz matrices

Graph of a



Eigenvalues of $T_{32}(a)$



First question: How many eigenvalues are in $[\alpha, \beta]$? (Szegő, 1920).

Second question: $\lambda_j^{(n)} \approx$?

Toeplitz matrices

Szegő's
limit theorem

Quantile function

Uniform
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Szegő's limit theorem (1920)

generating symbol
$a \in L^\infty([0, 2\pi], \mathbb{R})$

test function
$\varphi \in C(\mathbb{R})$

$$\frac{1}{n} \sum_{j=1}^{n} \varphi(\lambda_j^{(n)}) \longrightarrow \frac{1}{2\pi} \int_0^{2\pi} \varphi(a(\theta)) \, \mathrm{d}\theta$$

Avram–Parter limit theorem (1988)

an analogue of Szegő limit theorem for the singular values

generating symbol
$a \in L^\infty([0,2\pi],\mathbb{C})$

test function
$\varphi \in C(\mathbb{R})$

$$\frac{1}{n}\sum_{j=1}^{n}\varphi(s_j^{(n)}) \longrightarrow \frac{1}{2\pi}\int_0^{2\pi}\varphi(|a(\theta)|)\,\mathrm{d}\theta$$

Corollary from Szegő's limit theorem

distribution of the eigenvalues of Hermitian Toepltiz matrices

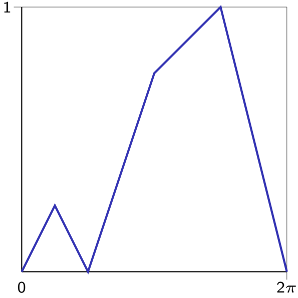
generating symbol
$a \in C([0, 2\pi], \mathbb{R})$

segment $[\alpha, \beta]$
$\alpha < \beta$

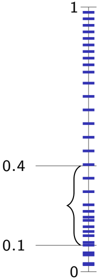
$$\frac{\#\{j:\ \alpha \le \lambda_j^{(n)} \le \beta\}}{n} \longrightarrow \frac{\mu\{\theta \in [0, 2\pi]:\ \alpha \le a(\theta) \le \beta\}}{2\pi}$$

Example to illustrate the corollary

Graph of a

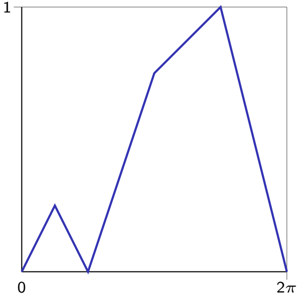


Eigenvalues of $T_{32}(a)$

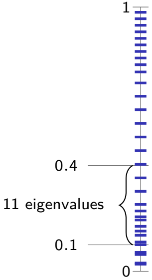


Example to illustrate the corollary

Graph of a



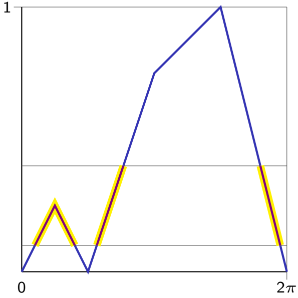
Eigenvalues of $T_{32}(a)$



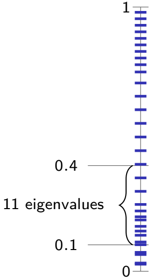
$$\frac{11}{32} \approx 0.344$$

Example to illustrate the corollary

Graph of a



Eigenvalues of $T_{32}(a)$



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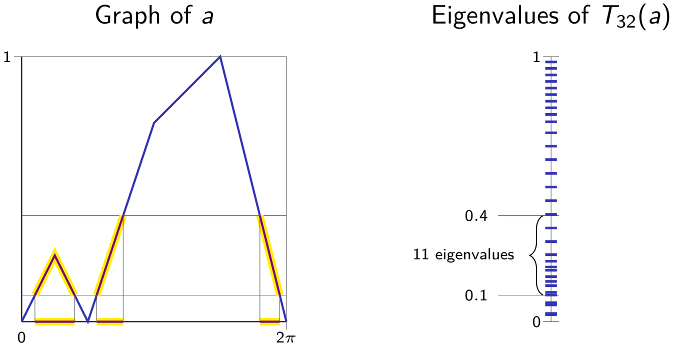
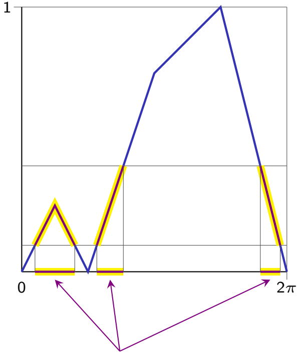
Example to illustrate the corollary



$$\frac{11}{32} \approx 0.344$$

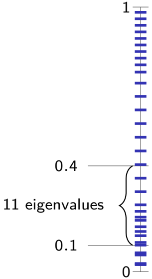
Example to illustrate the corollary

Graph of a



$$\frac{\mu\{\theta:\ \ 0.1 \le a(\theta) \le 0.4\}}{2\pi} = 0.325$$

Eigenvalues of $T_{32}(a)$



$$\frac{11}{32} \approx 0.344$$

Szegő found an approximate answer to the first question (the number of the eigenvalues on any segment $[\alpha, \beta]$).

The second question is open:

$$\lambda_j^{(n)} \approx ?$$

Corollary by Trench (2012): the reordering idea

$$a \in C([0, 2\pi], \mathbb{R})$$

↓

$$\frac{1}{n} \sum_{j=1}^{n} \left| \lambda_j^{(n)} - v_j^{(n)} \right| \longrightarrow 0$$

where $v_1^{(n)}, \ldots, v_n^{(n)}$ are the values $a\left(\frac{2\pi}{n}\right), a\left(\frac{4\pi}{n}\right), \ldots, a\left(\frac{2\pi n}{n}\right)$

written in the ascending order: $v_1^{(n)} \le \cdots \le v_n^{(n)}$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Quantile function of a list of numbers

118 100 195 166 164 123 102 172 164 117

The same numbers in the ascending order:

100 102 117 118 123 164 164 166 172 195

$$\text{QuantileFunction}(1/3) = 118$$

because 118 is the minimal number v
such that at least $1/3$ of the elements are less or equal to v.

Quantile function associated to $a \in L^\infty([0, 2\pi], \mathbb{R})$

$F_a :=$ the cumulative distribution function of a:

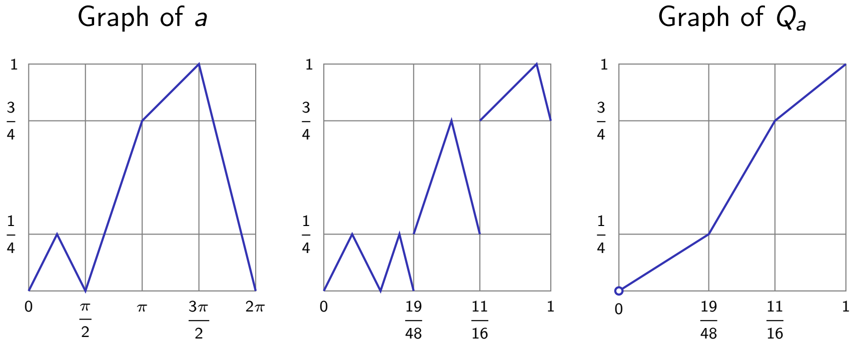
$$F_a(v) := \frac{1}{2\pi}\,\mu\,\{\theta \in [0, 2\pi]\colon\ a(\theta) \le v\}, \qquad v \in \mathbb{R}.$$

$Q_a :=$ the corresponding quantile function:

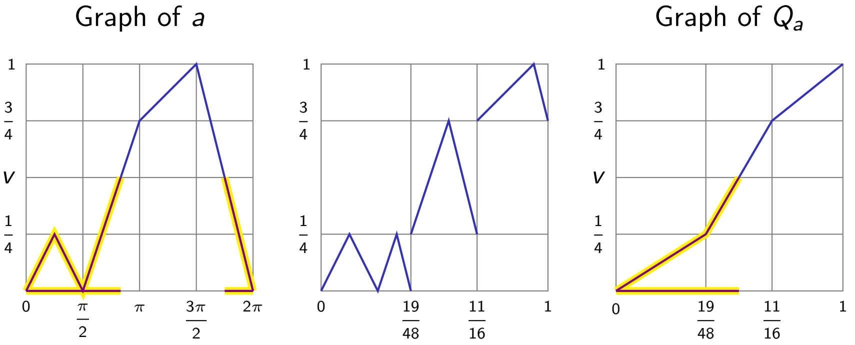
$$Q_a(p) := \inf\{v \in \mathbb{R}\colon\ F_a(v) \ge p\}, \qquad p \in (0, 1].$$

Q_a increases and has the same distribution as a.

Construction of the quantile function associated to a piecewise-linear generating symbol



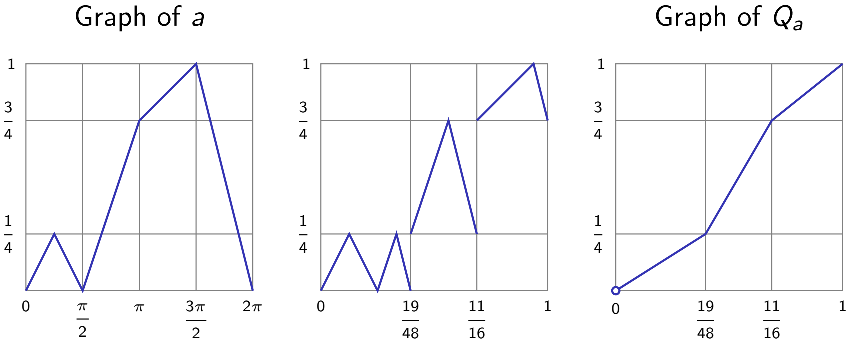
Construction of the quantile function associated to a piecewise-linear generating symbol



a and Q_a are identically distributed:

$$\frac{1}{2\pi}\mu\{\theta \in [0,2\pi]:\ \ a(\theta) \le v\} = \mu\{p \in (0,1]:\ \ Q_a(p) \le v\}$$

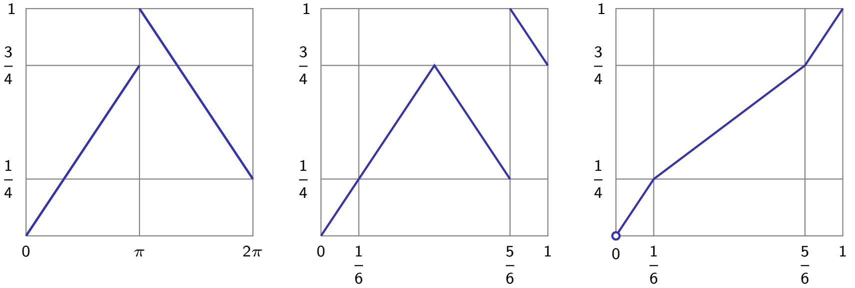
Construction of the quantile function associated to a piecewise-linear generating symbol



$$a \xrightarrow{\text{reordering in the Lebesgue-style}} Q_a$$

Q_a may be continuous even when a is not

Graph of a Graph of Q_a



Criterion of continuity of the quantile function

Let $a\colon [0, 2\pi] \to \mathbb{R}$ be a measurable function.
$\mathcal{R}(a) :=$ the essential range of a.
Then the following conditions are equivalent.

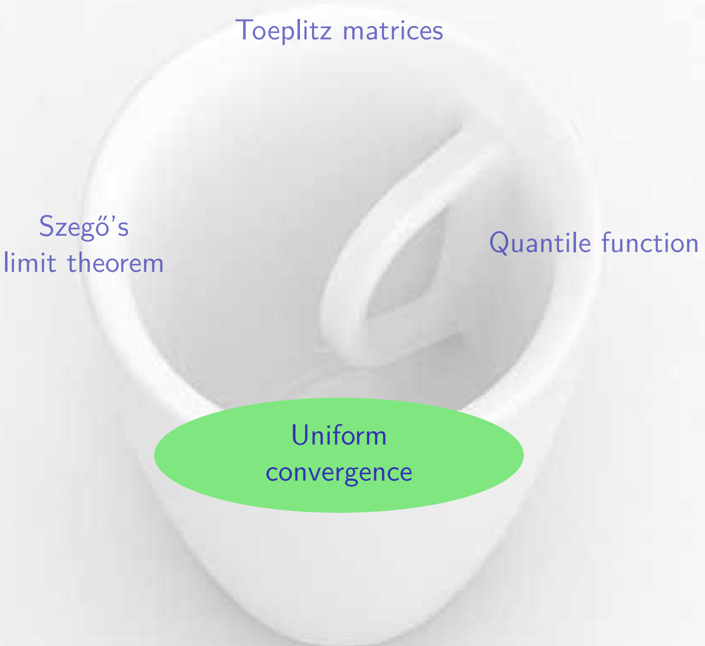
$\mathcal{R}(a)$ is connected

$\Updownarrow$

F_a is strictly increasing

$\Updownarrow$

Q_a is continuous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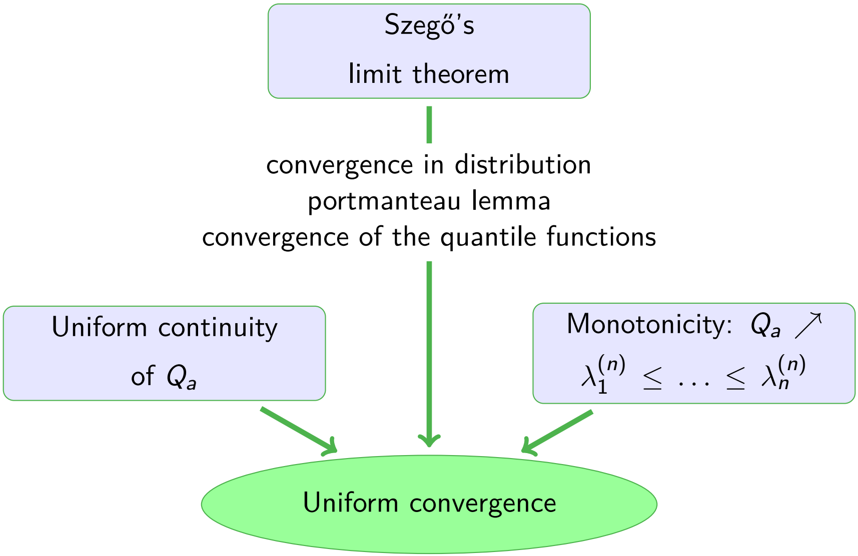
Main result: uniform convergence of the eigenvalues

$a \in L^\infty([0, 2\pi], \mathbb{R})$

$\mathcal{R}(a)$ is connected

$$\max_{1 \le j \le n} \left| \lambda_j^{(n)} - Q_a(j/n) \right| \longrightarrow 0$$

Idea of the proof



Uniform convergence of the singular values

(our corollary from Avram–Parter theorem)

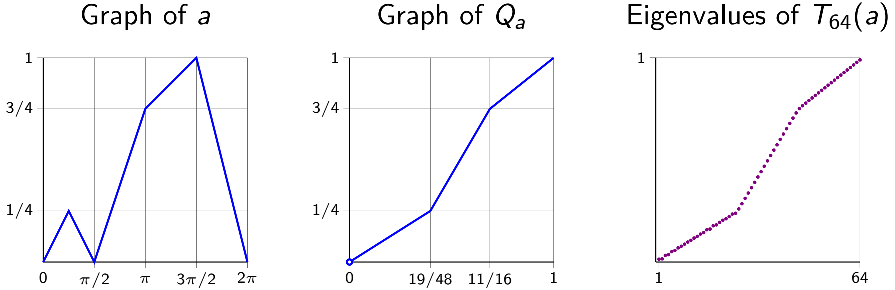
$$a \in L^\infty([0, 2\pi], \mathbb{C})$$

$$\mathcal{R}(|a|) = [0, \|a\|_\infty]$$

$$\max_{1 \le j \le n} \left| s_j^{(n)} - Q_{|a|}(j/n) \right| \longrightarrow 0$$

First example

continuous piecewise-linear generating symbol



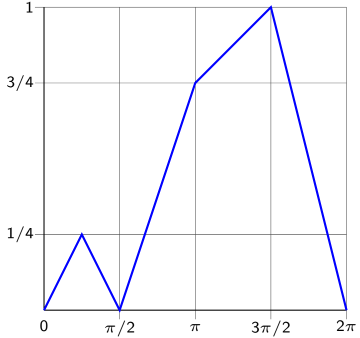
Every eigenvalue $\lambda_j^{(n)}$ is shown as a point $\left(\frac{j}{n},\ \lambda_j^{(n)}\right)$.

The third picture mimics the second one.

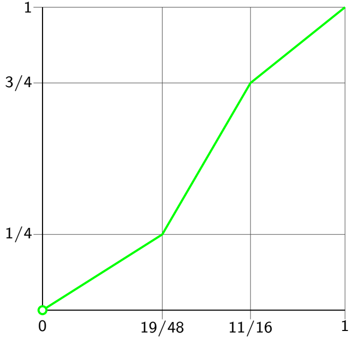
First example

continuous piecewise-linear generating symbol

Graph of a



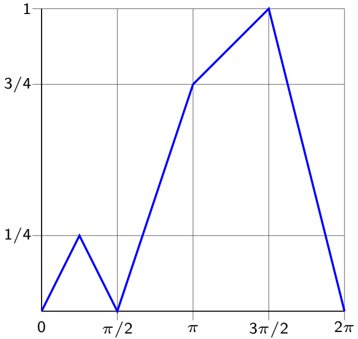
Graph of Q_a



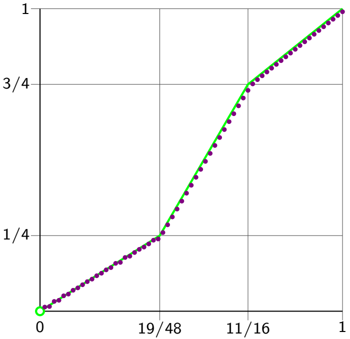
First example

continuous piecewise-linear generating symbol

Graph of a



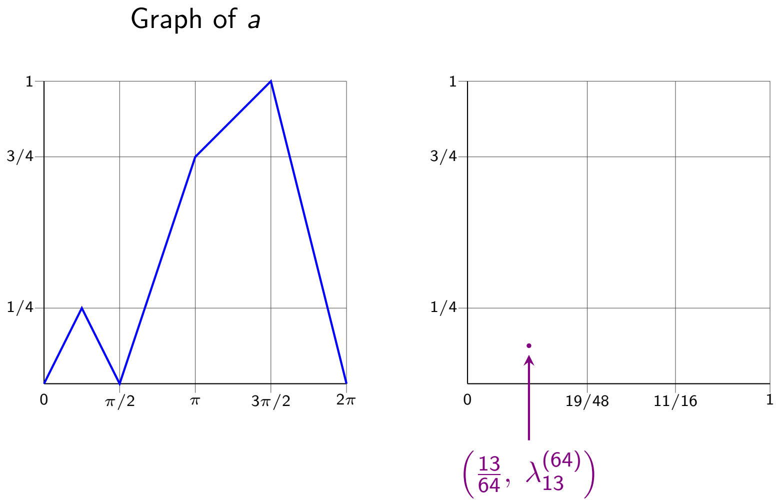
Graph of Q_a



and the points $(j/n, \lambda_j^{(n)})$

First example

continuous piecewise-linear generating symbol



First example

continuous piecewise-linear generating symbol

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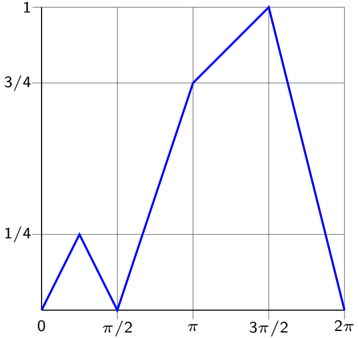
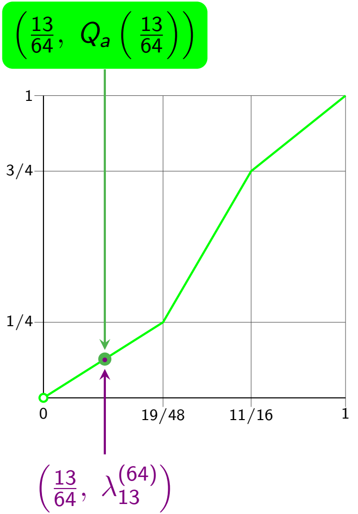
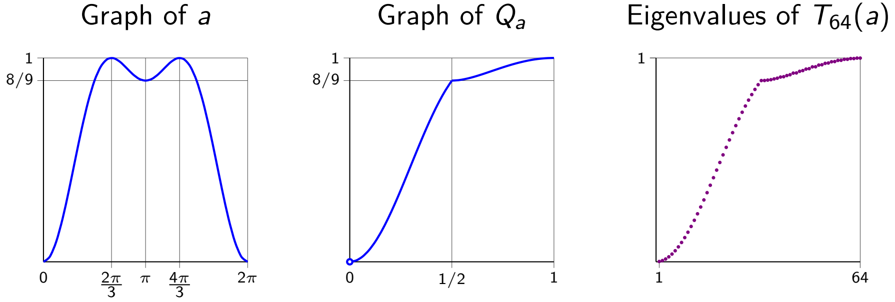




Second example

trigonometric polynomial $a(\theta) = \frac{2}{3} - \frac{4}{9}\cos(\theta) - \frac{2}{9}\cos(2\theta)$

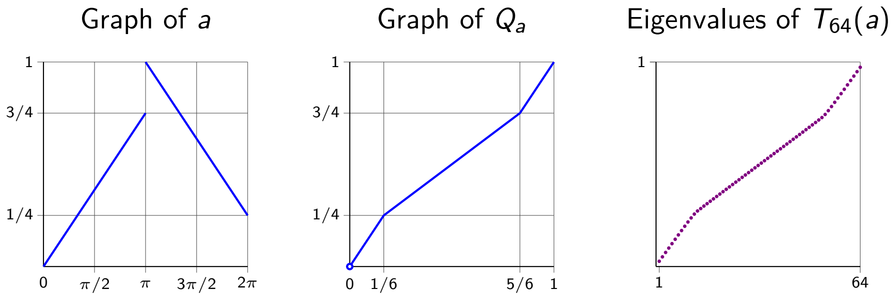


If the generating symbol is a trigonometric polynomial, then the maximal error is of the order $O(1/n)$:

$$\exists C > 0 \qquad \max_{1 \le j \le n} \left| \lambda_j^{(n)} - Q_a(j/n) \right| \le \frac{C}{n}$$

Third example

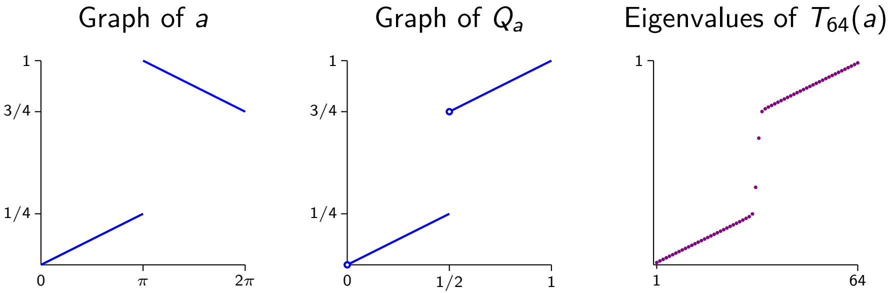
a is not continuous, but $\mathcal{R}(a)$ is connected



In this example, $\lambda_j^{(n)}$ is also uniformly approximated by $Q_a(j/n)$ as $n \to \infty$.

Fourth example

If $\mathcal{R}(a)$ is not connected, then the uniform convergence fails



In this example, $\lambda^{(n)}_{\lfloor n/2 \rfloor}$ can not be approximated by values of Q_a.

Summary

The Szegő limit theorem combined with the notion of quantile function yields the main term of the individual asymptotics of the eigenvalues:

$$\lambda_j^{(n)} \approx Q_a\left(\frac{j}{n}\right)$$

assuming that $a \in L^\infty([0, 2\pi], \mathbb{R})$ and $\mathcal{R}(a)$ is connected.

Summary

The Szegő limit theorem combined with the notion of quantile function yields the main term of the individual asymptotics of the eigenvalues:

$$\lambda_j^{(n)} \approx \; Q_a\left(\frac{j}{n}\right)$$

assuming that $a \in L^\infty([0, 2\pi], \mathbb{R})$ and $\mathcal{R}(a)$ is connected.

Thanks for attention!