Vertical Toeplitz operators on the upper half-plane and very slowly oscillating functions

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joint work with
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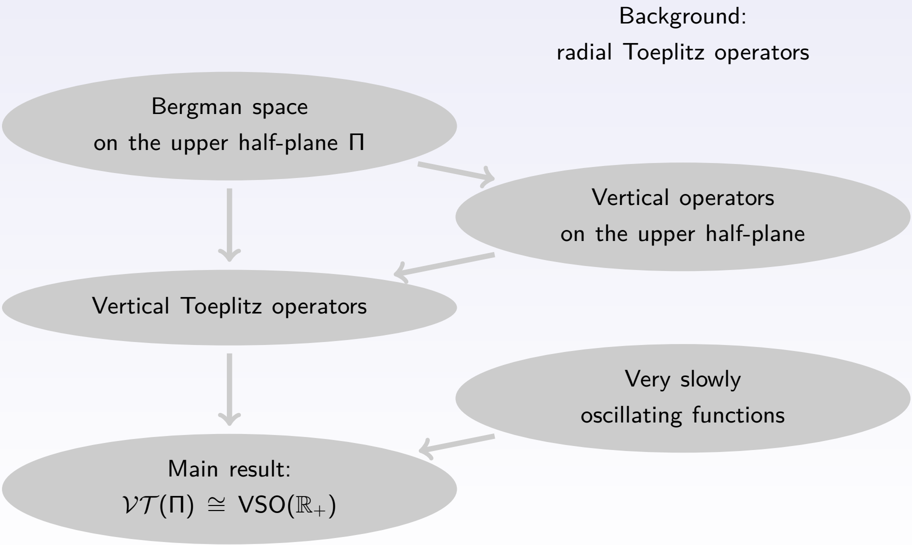
Outline


Background:
radial Toeplitz operators
Bergman space
on the upper half-plane Π
Vertical operators
on the upper half-plane
Vertical Toeplitz operators
Very slowly
oscillating functions
Main result:
$\mathcal{VT}(\Pi) \cong \mathrm{VSO}(\mathbb{R}_+)$

Background: radial, vertical and angular Toeplitz opeators

Korenblum and Zhu (1995):
Toeplitz opeators with radial symbols acting in $\mathcal{A}^2(\mathbb{D})$ are diagonal with respect to the canonical (monomial) basis, therefore they generate a commutative C^* algebra.

These operators have been studied by many authors including Grudsky and Vasilevski (2001), Zorboska (2003), Suárez (2008).

Vasilevski found another two commutative C^*-algebras generated by Toeplitz operators on the upper half-plane:

- "vertical" Toeplitz operators with symbols depending on $\mathrm{Im}(z)$;
- "angular" Toeplitz operators with symbols depending on $\arg(z)$.

In this talk we present some results about vertical Toeplitz operators, mentioning similar results for the radial Toeplitz operators.

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Bergman space on a domain $\Omega \subset \mathbb{C}$

Let Ω be an open connected subset of $\mathbb{C}$
considered with the usual Lebesgue measure μ.

In this talk Ω will be the unit disk $\mathbb{D}$ or the upper half-plane Π:

$$\Pi := \{z \in \mathbb{C}:\ \ \mathrm{Im}(z) > 0\}.$$

$L^2(\Omega) :=$ square-integrable functions on Ω.

$\mathcal{A}(\Omega) :=$ holomorphic functions on Ω.

$\mathcal{A}^2(\Omega) \ := \ L^2(\Omega) \cap \mathcal{A}(\Omega)$ is called the Bergman space over Ω.

$\mathcal{A}^2(\Omega)$ is a closed subspace of $L^2(\Omega)$. Therefore it is a Hilbert space.

Bergman kernel and Bergman projection

For every $z \in \Omega$ denote by ev_z the evaluation functional at the point z:

$$\mathrm{ev}_z \colon \mathcal{A}^2(\Omega) \to \mathbb{C}, \qquad \mathrm{ev}_z(f) := f(z).$$

It is possible to show that this evaluation functional ev_z is bounded.
By Riesz–Fréchet theorem, there exists a function $K_{\Omega,z} \in \mathcal{A}^2(\Omega)$ such that

$$\forall f \in \mathcal{A}^2(\Omega) \qquad \langle f, K_{\Omega,z} \rangle = f(z). \tag{RP}$$

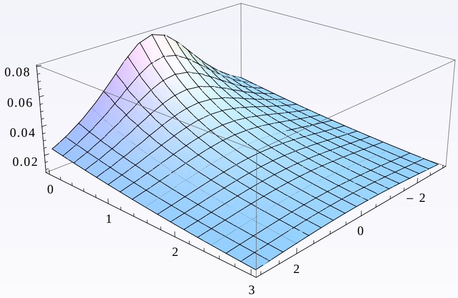
$K_{\Omega,z}$ is called the Bergman kernel corresponding to Ω and z.
(RP) is called the reproducing property of the Bergman kernel.

Orthogonal Bergman projection P_Ω of $L^2(\Omega)$ over $\mathcal{A}^2(\Omega)$:

$$\forall z \in \Omega \qquad \forall f \in L^2(\Omega) \qquad (P_\Omega f)(z) = \langle f, K_{\Omega,z} \rangle.$$

Bergman kernel corresponding to the upper half-plane

$$K_{\Pi,z}(w) = \frac{1}{\pi(w - \overline{z})^2}.$$



The picture shows $|K_{\Pi,2\,\mathrm{i}}(w)|$ for $-3 \leq \mathrm{Re}(w) \leq 3$, $0 < \mathrm{Im}(w) < 3$.

Toeplitz operators generated by bounded functions

If $g \in L^\infty(\Omega)$, denote by M_g the multiplication by g in $L^2(\Omega)$:

$$M_g \colon L^2(\Omega) \to L^2(\Omega), \qquad M_g(f) := fg.$$

The Toeplitz operator with generating symbol g can be defined as the compression of M_g to the subspace $\mathcal{A}^2(\Omega)$:

$$T_g f := P_\Omega(fg).$$

T_g can be written in the following integral form using the Bergman kernel:

$$(T_g f)(z) = \langle f\, g, K_{\Omega,z} \rangle = \int_\Omega f(w) g(w) \overline{K_{\Omega,z}(w)}\, d\mu(w).$$

General properties of Toeplitz operators on the Bergman space have been studied by Ahern, Axler, Coburn, Čučković, Rao, Zheng, Zhu, etc.

Berezin transform of bounded linear operators

Given a bounded linear operator $S\colon \mathcal{A}^2(\Omega) \to \mathcal{A}^2(\Omega)$,
its Berezin transform $\mathcal{B}(S)$ is defined by

$$\mathcal{B}(S)(w) := \frac{\langle S K_{\Omega,w}, K_{\Omega,w} \rangle}{\langle K_{\Omega,w}, K_{\Omega,w} \rangle}.$$

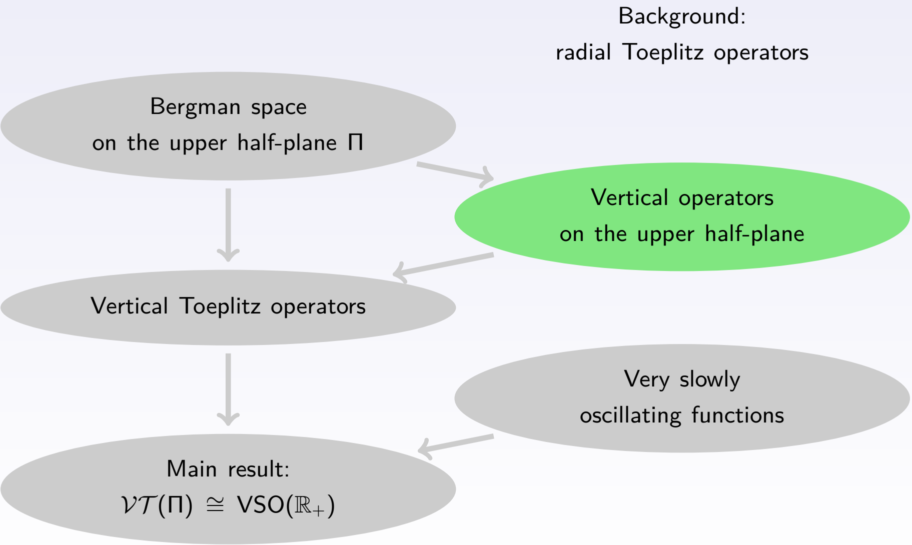
Note that $\mathcal{B}(S)$ is a $\mathbb{R}^2$-analytic function $\Omega \to \mathbb{C}$.

Stroethoff (1997) proved that the correspondence $S \mapsto \mathcal{B}(S)$ is injective.

The Berezin transform of a function $g \in L^\infty(\Omega)$ is defined by

$$\mathcal{B}(g)(w) := \frac{\langle T_g K_{\Omega,w}, K_{\Omega,w} \rangle}{\langle K_{\Omega,w}, K_{\Omega,w} \rangle} = \frac{\langle g\, K_{\Omega,w}, K_{\Omega,w} \rangle}{\langle K_{\Omega,w}, K_{\Omega,w} \rangle}.$$

Outline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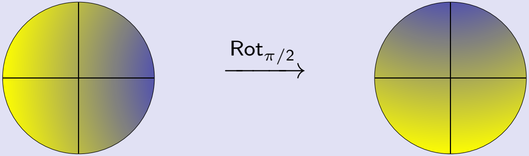
Background: rotations in $\mathcal{A}^2(\mathbb{D})$

Given $\theta \in \mathbb{R}$, denote by Rot_θ the rotation operator:

$$\mathrm{Rot}_\theta \colon \mathcal{A}^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D}), \qquad \mathrm{Rot}_\theta(f)(z) := f(\mathrm{e}^{-\mathrm{i}\,\theta} z).$$

Example

$$f(z) = \frac{1}{2 - z}, \qquad (\mathrm{Rot}_{\pi/2}\, f)(z) = \frac{1}{2 + \mathrm{i}\, z}.$$



Background: radial operators on the unit disk

A bounded linear operator $S\colon \mathcal{A}^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D})$ is called radial if it commutes with the rotation operators:

$$\forall \theta \in \mathbb{R} \qquad \operatorname{Rot}_\theta S = S \operatorname{Rot}_\theta .$$

Criterion of radial operators (Zorboska, 2003)

Given a bounded linear operator $S\colon \mathcal{A}^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D})$, the following conditions are equivalent:

(a) *S is radial.*

(b) *S is diagonal with respect to the canonical basis in $\mathcal{A}^2(\mathbb{D})$:*

$$\exists \lambda = (\lambda_j)_{j=0}^{\infty} \in \ell^\infty(\mathbb{N}_0) \qquad \forall j \in \mathbb{N}_0 \qquad S e_j = \lambda_j e_j .$$

(c) *The Berezin transform of S is a radial function:*

$$\forall z \in \mathbb{D} \qquad \mathcal{B}(S)(z) = \mathcal{B}(S)(|z|).$$

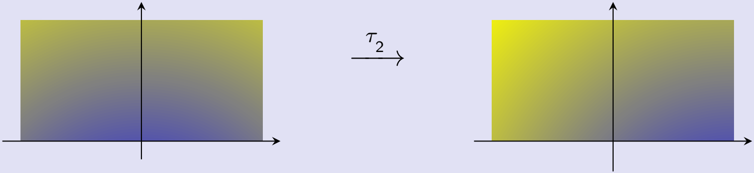
Horizontal translations of the upper half-plane

Given $h \in \mathbb{R}$, denote by τ_h the horizontal translation operator:

$$\tau_h \colon \mathcal{A}^2(\Pi) \to \mathcal{A}^2(\Pi), \qquad (\tau_h f)(w) \coloneqq f(w-h).$$

Example

$$f(w) = \frac{1}{(w+2\,\mathrm{i})^2}, \qquad (\tau_2 f)(z) = \frac{1}{(w-2+2\,\mathrm{i})^2}.$$



The operator τ_h is unitary, with $\tau_h^* = \tau_h^{-1} = \tau_{-h}$.

Vertical operators in $\mathcal{A}^2(\Pi)$

We say that a bounded linear operator $S\colon \mathcal{A}^2(\Pi) \to \mathcal{A}^2(\Pi)$ is invariant under horizontal translations or vertical if

$$\forall h \in \mathbb{R} \qquad \tau_h S = S \tau_h.$$

Isomorphism $R\colon \mathcal{A}^2(\Pi) \to L^2(\mathbb{R}_+)$

Vasilevski (1999) constructed an isometric isomorphism

$$R\colon \mathcal{A}^2(\Pi) \to L^2(\mathbb{R}_+),$$

$$(R\varphi)(x) := \frac{\sqrt{x}}{\sqrt{\pi}} \int_{\Pi} \varphi(w)\, \mathrm{e}^{-\mathrm{i}\,\overline{w}x}\, d\mu(w).$$

The operator R is unitary, and its inverse $R^*\colon L^2(\mathbb{R}_+) \to \mathcal{A}^2(\Pi)$ is given by

$$(R^* f)(z) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}_+} \sqrt{\xi}\, f(\xi)\, \mathrm{e}^{\mathrm{i}\,z\xi}\, d\xi.$$

The operators R and R^* serve us to diagonalize the vertical operators.

Criterion of vertical operators

Theorem

Let $S\colon \mathcal{A}^2(\Pi) \to \mathcal{A}^2(\Pi)$ *be a bounded linear operator.*
Then the following conditions are equivalent:

(a) *S is vertical.*

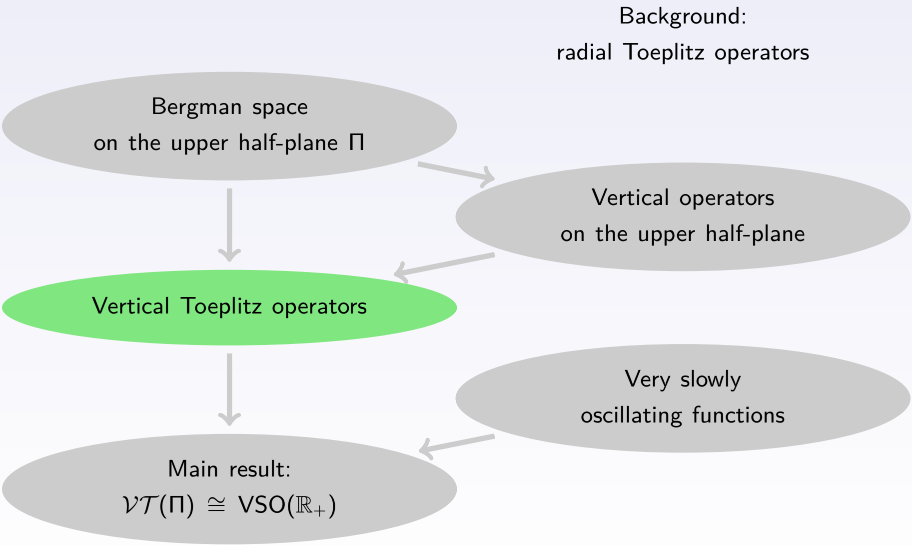
(b) *S is diagonalized by the unitary operator R:*

$$\exists \sigma \in L^\infty(\mathbb{R}_+) \qquad RSR^* = M_\sigma.$$

(c) *The Berezin transform of S depends on the imaginary part only:*

$$\mathcal{B}(S)(u + \mathrm{i}\, v) = \mathcal{B}(S)(\mathrm{i}\, v) \qquad \forall u \in \mathbb{R},\ v > 0.$$

Outline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Background: radial Toeplitz operators

Theorem (Korenblum and Zhu, 1995)

Let $a \in L^\infty([0,1))$ and g be the radial extension of a:

$$\forall z \in \mathbb{D} \qquad g(z) := a(|z|).$$

Then T_g is diagonal with respect to the canonical basis $(e_j)_{j=0}^\infty$ in $\mathcal{A}^2(\mathbb{D})$:

$$T_g e_j = \lambda_a(j) e_j, \qquad \text{where} \qquad \lambda_a(j) = (j+1)\int_0^1 a(\sqrt{r}\,)\, r^j\, dr.$$

Theorem (Zorboska, 2003)

Let $g \in L^\infty(\mathbb{D})$. Then T_g *is radial* $\iff$ g *is radial.*

Criterion of vertical Toeplitz operators

Let $g \in L^\infty(\Pi)$. We say that g is *vertical* if there exists a function $a \in L^\infty(\mathbb{R}_+)$ such that

$$g(w) = a(\operatorname{Im}(w)) \qquad \text{a.e.} \quad w \in \Pi.$$

Theorem

Let $g \in L^\infty(\Pi)$. Then T_g *is vertical* $\iff$ g *is vertical.*

The implication $\Longleftarrow$ was proved by Vasilevski in 2003.

Diagonalization of vertical Toeplitz operators

Recall the unitary operator

$$R\colon \mathcal{A}^2(\Pi) \to L^2(\mathbb{R}_+),$$

$$(R\varphi)(x) := \frac{\sqrt{x}}{\sqrt{\pi}} \int_\Pi \varphi(w)\, \mathrm{e}^{-\mathrm{i}\,\overline{w}x}\, d\mu(w).$$

Theorem (Vasilevski, 2003)

Let $a \in L^\infty(\mathbb{R}_+)$. Denote by $\tilde{a}$ the vertical extension of a:

$$\forall w \in \Pi \qquad \tilde{a}(w) := a(\mathrm{Im}(w)).$$

Then $RT_{\tilde{a}}R^* = M_{\gamma_a}$, where

$$\gamma_a(x) = x \int_{\mathbb{R}_+} a(v/2)\, \mathrm{e}^{-xv}\, dv.$$

Passing from operators to their "spectral functions"

$T_{\tilde{a}}$ γ_a

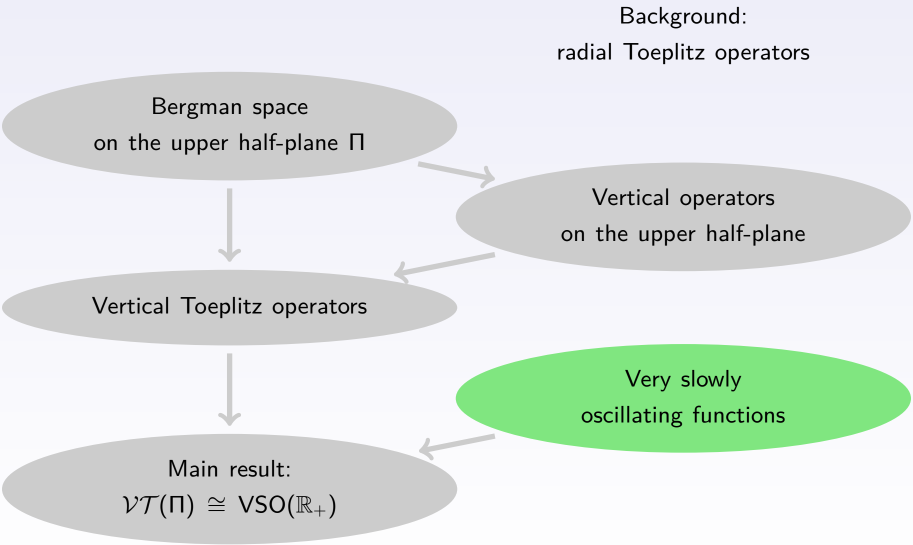
$\Xi := \{ T_{\tilde{a}} : \ a \in L^\infty(\mathbb{R}_+)\}$ $\Gamma := \{\gamma_a : \ a \in L^\infty(\mathbb{R}_+)\}$

$\mathcal{VT}(\Pi) := C^*\text{-algebra}(\Xi)$ $\mathcal{A} := C^*\text{-algebra}(\Gamma)$

The C^*-algebras $\mathcal{VT}(\Pi)$ and $\mathcal{A}$ are isometrically isomorphic.
In particular, $\mathcal{VT}(\Pi)$ is commutative.

$\mathcal{A}$ is a C^*-subalgebra of $C_b(\mathbb{R}_+)$. Problem: describe $\mathcal{A}$.

Outline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Background: very slowly oscillating sequences

This class of sequences was introduced by Robert Schmidt in 1925:

$$\mathsf{VSO}(\mathbb{N}_0) := \Big\{\lambda \in \ell^\infty(\mathbb{N}_0) \colon \quad \lim_{\frac{j+1}{k+1}\to 1} \big|\lambda_j - \lambda_k\big| = 0\Big\}.$$

Robert Schmidt called them “slowly oscillating sequences” (“langsam oszillierenden Folgen”), we prefer to say “very slowly”.

This class of sequences has many applications in Tauberian theory.

Subclasses of $\mathrm{VSO}(\mathbb{N}_0)$

Converging sequences

If $\lambda = (\lambda_j)_{j=0}^{\infty}$ has a finite limite, then $\lambda \in \mathrm{VSO}(\mathbb{N}_0)$.

Class $d_1(\mathbb{N}_0)$

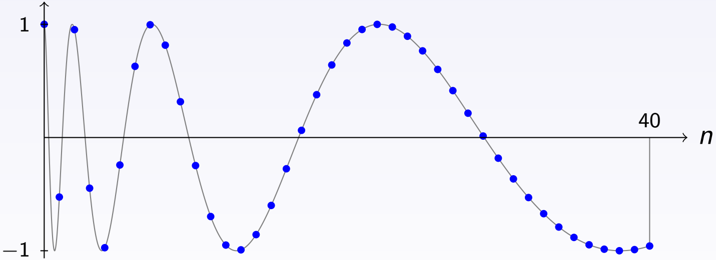
$$d_1(\mathbb{N}_0) := \left\{ \lambda \in \ell^\infty(\mathbb{N}_0) \colon \ \sup_{j \in \mathbb{N}_0} \Big((j+1) \big| \lambda_{j+1} - \lambda_j \big| \Big) < +\infty \right\}.$$

It is easy to see that $d_1(\mathbb{N}_0) \subset \mathrm{VSO}(\mathbb{N}_0)$.

An example of a sequence in $\mathrm{VSO}(\mathbb{N}_0)$

$$\lambda_n = \cos\Big(6\log(n+1)\Big).$$

This sequence is not converging, but belongs to $\mathrm{VSO}(\mathbb{N}_0)$.



Very slowly oscillating functions, first definition

We shall give three equivalent definitions of $\mathrm{VSO}(\mathbb{R}_+)$.

Let $\sigma\colon \mathbb{R}_+ \to \mathbb{C}$ be a bounded function.
We say that σ is very slowly oscillating if

$$\lim_{\frac{x}{y}\to 1} |\sigma(x) - \sigma(y)| = 0,$$

i.e.,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, y > 0 \quad \left(\left|\frac{x}{y} - 1\right| < \delta \quad \Longrightarrow \quad |\sigma(x) - \sigma(y)| < \varepsilon \right).$$

Very slowly oscillating functions, second definition

Let $\sigma\colon \mathbb{R}_+ \to \mathbb{C}$ be a bounded function.
σ is called very slowly oscillating if the composition

$$\sigma \circ \exp\colon \mathbb{R} \to \mathbb{C}$$

is uniformly continuous with respect to the usual metric d on $\mathbb{R}$.

Very slowly oscillating functions, third definition

"Logarithmic metric" in $\mathbb{R}_+$:

$$\rho(x,y) \coloneqq \Bigl|\log(x) - \log(y)\Bigr| = \Bigl|\log\frac{x}{y}\Bigr|.$$

Modulus of continuity of a function $\sigma\colon \mathbb{R}_+ \to \mathbb{C}$ with respect to ρ:

$$\omega_{\rho,\sigma}\colon\ [0,+\infty) \to [0,+\infty],$$

$$\omega_{\rho,\sigma}(\delta) \coloneqq \sup\bigl\{|\sigma(x)-\sigma(y)|\colon\ \rho(x,y) \le \delta\bigr\}.$$

Very slowly oscillating functions
are bounded uniformly continuous functions $(\mathbb{R}_+,\rho) \to \mathbb{C}$:

$$\mathrm{VSO}(\mathbb{R}_+) \coloneqq \Bigl\{\sigma \in L^\infty(\mathbb{R}_+)\colon\ \lim_{\delta\to 0^+} \omega_{\rho,\sigma}(\delta) = 0\Bigr\}.$$

Some simple properties of $\mathrm{VSO}(\mathbb{R}_+)$

- $\mathrm{VSO}(\mathbb{R}_+)$ is closed C^*-subalgebra of the C^*-algebra $C_b(\mathbb{R}_+)$.

- If $\sigma\colon \mathbb{R}_+ \to \mathbb{C}$ is continuous on $\mathbb{R}_+$
  and have finite limits at 0 and $+\infty$,
  then $\sigma \in \mathrm{VSO}(\mathbb{R}_+)$:

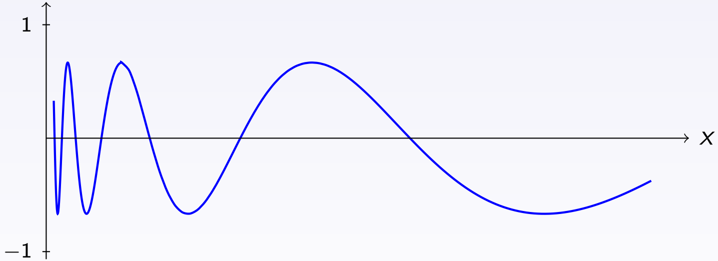
$$C([0,+\infty]) \subset \mathrm{VSO}(\mathbb{R}_+).$$

Example of a function in $\mathrm{VSO}(\mathbb{R}_+)$

The function

$$\sigma(x) = \cos(5\log(x))$$

does not have limits at 0 and $+\infty$, but belongs to $\mathrm{VSO}(\mathbb{R}_+)$.



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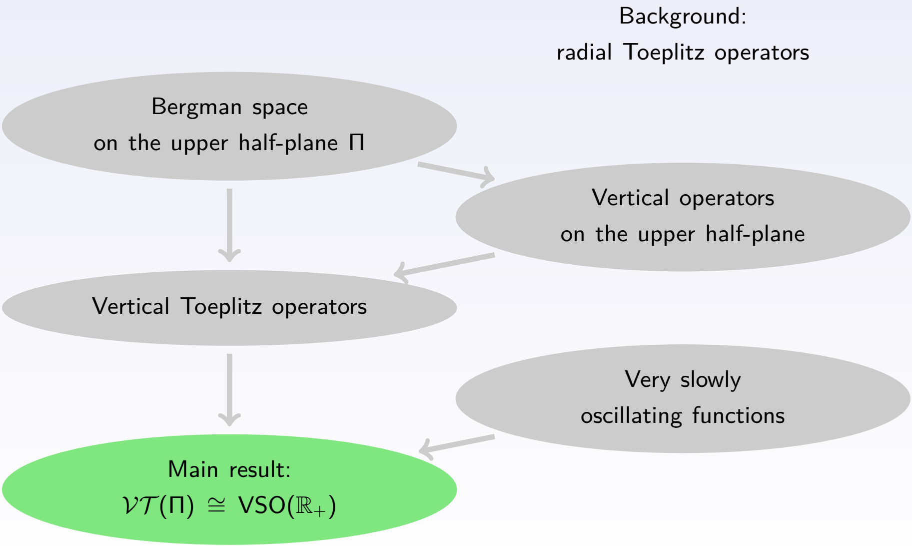
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Background: radial case

Denote by Λ the set of the sequences of the eigenvalues corresponding to the radial Toeplitz operators.

Theorem (Daniel Suárez, 2008)

The C^*-algebra generated by Λ coincides with the closure of Λ in $\ell^\infty(\mathbb{N}_0)$ and is equal to the closure of $d_1(\mathbb{N}_0)$ in $\ell^\infty(\mathbb{N}_0)$, where

$$d_1(\mathbb{N}_0) = \Big\{ y \in \ell^\infty(\mathbb{N}_0) : \quad \sup_{j \geq 0} \Big((j+1) \, |y_{j+1} - y_j| \Big) < +\infty \Big\}.$$

The proof of Suárez is based on the so-called n-Berezin transform.

Corollary (Grudsky, Maximenko, Vasilevski, 2013)

The C^*-algebra generated by Λ is $\mathrm{VSO}(\mathbb{N}_0)$.

Main result about the vertical Toeplitz operators

Recall that Γ is the set of the "spectral functions" γ_a corresponding to the vertical Toeplitz operators:

$$\Gamma = \{\gamma_a\colon\ a \in L^\infty(\mathbb{R}_+)\}, \quad \text{where} \quad \gamma_a(x) = x \int_{\mathbb{R}_+} a(v/2)\, \mathrm{e}^{-xv}\, dv.$$

Theorem

Γ is a dense subset of $\mathrm{VSO}(\mathbb{R}_+)$.

Corollary 1: $C^*\text{-alg}(\Gamma) \quad = \quad$ *closure of* Γ *in* $C_b(\mathbb{R}_+) \quad = \quad \mathrm{VSO}(\mathbb{R}_+)$.

Corollary 2: C^*-algebra generated by vertical Toeplitz operators is isomorphic to $\mathrm{VSO}(\mathbb{R}_+)$.

Idea of the proof: we expressed γ_a in terms of the Mellin convolution and constructed a special Dirac sequence in $L^1(\mathbb{R}_+, dv/v)$.

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