Radial Toeplitz operators on the unit ball in $\mathbb{C}^n$ and slowly oscillating sequences

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joint work with
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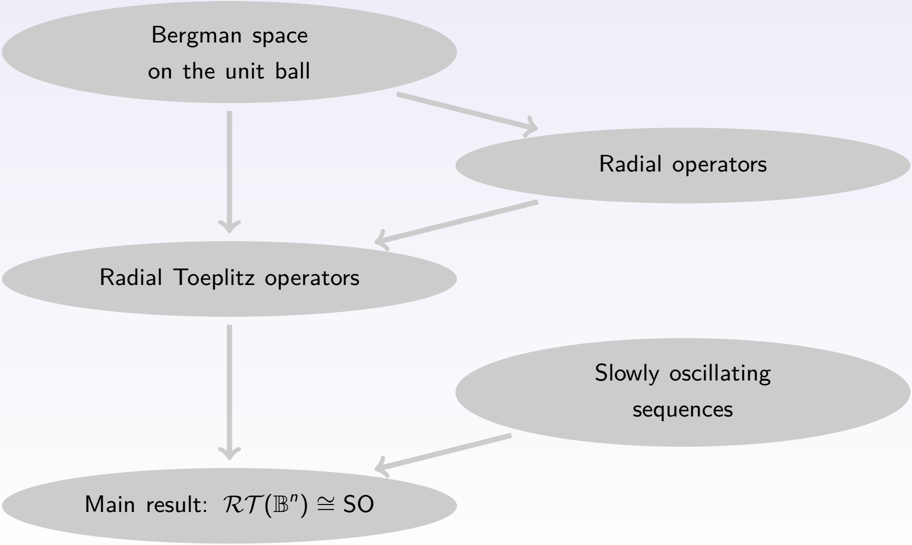
Outline



Bergman space
on the unit ball
Radial operators
Radial Toeplitz operators
Slowly oscillating
sequences
Main result: $\mathcal{RT}(\mathbb{B}^n) \cong \mathrm{SO}$

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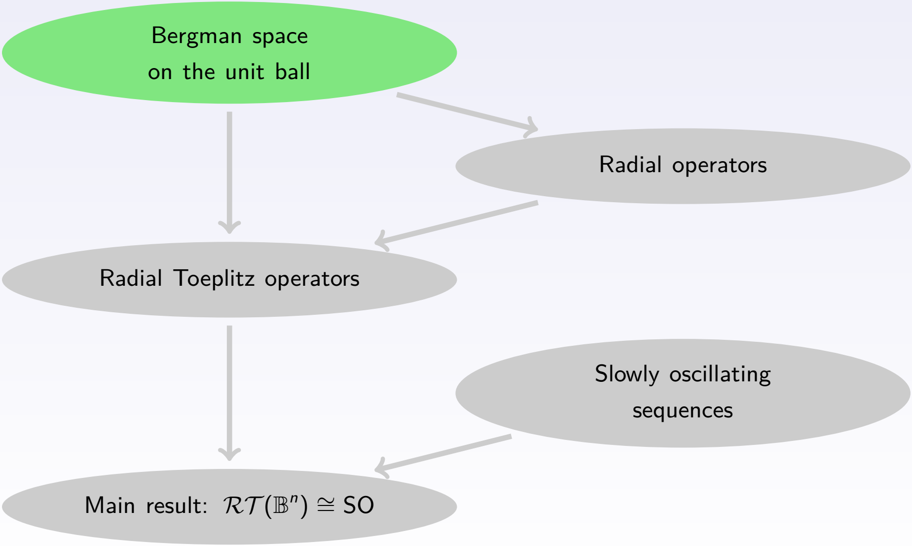
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Notation: unit ball and normalized Lebesgue measure

Usual inner product and Euclidean norm in $\mathbb{C}^n$:

$$\langle z, w \rangle := \sum_{j=1}^{n} z_j \overline{w_j}, \qquad \|z\| := \sqrt{\langle z, z \rangle}.$$

Unit ball in $\mathbb{C}^n$:

$$\mathbb{B}^n := \{z \in \mathbb{C}^n : \ \|z\| < 1\}.$$

dv := the Lebesgue measure in $\mathbb{C}^n = \mathbb{R}^{2n}$ normalized such that $v(\mathbb{B}^n) = 1$.

$$dv = \frac{n!}{\pi^n} dx_1 dy_1 \cdots dx_n dy_n.$$

Bergman space on the unit ball

$L^2(\mathbb{B}^n) := L^2(\mathbb{B}^n, dv) :=$ the space of square integrable functions, provided with the usual inner product:

$$\langle f, g \rangle := \int_{\mathbb{B}^n} f\,\overline{g}\,dv, \qquad \|f\|_2 := \sqrt{\langle f, f \rangle}.$$

$\mathcal{A}(\mathbb{B}^n) :=$ the set of holomorph functions $\mathbb{B}^n \to \mathbb{C}$.

$\mathcal{A}^2(\mathbb{B}^n) \;:=\; L^2(\mathbb{B}^n) \cap \mathcal{A}(\mathbb{B}^n)$, is called the Bergman space over $\mathbb{B}^n$.

$\mathcal{A}^2(\mathbb{B}^n)$ is a closed subspace of $L^2(\mathbb{B}^n)$ and therefore is a Hilbert space.

Canonical basis in $\mathcal{A}^2(\mathbb{B}^n)$

$\mathbb{N} := \{0, 1, 2, \ldots\}$.

Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, recall usual notation for the sum of components, the factorial and the power:

$$|\alpha| := \sum_{j=1}^{n} \alpha_j, \qquad \alpha! := \prod_{j=1}^{n} \alpha_j!, \qquad z^\alpha := \prod_{j=1}^{n} z_j^{\alpha_j}.$$

Denote by e_α, where $\alpha \in \mathbb{N}^n$, the normalized monomials:

$$e_\alpha(z) := \sqrt{\frac{(n + |\alpha|)!}{n!\,\alpha!}}\, z^\alpha.$$

The family $(e_\alpha)_{\alpha \in \mathbb{N}^n}$ is an orthonormal basis in $\mathcal{A}^2(\mathbb{B}^n)$.

Bergman kernel and Bergman projection

For every $z \in \mathbb{B}^n$, the Bergman kernel at the point z is the function

$$K_z(w) = \sum_{\alpha \in \mathbb{N}^n} \overline{e_\alpha(z)} e_\alpha(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+1}}.$$

Reproduction property:

$$\forall f \in \mathcal{A}^2(\mathbb{B}^n) \qquad \forall z \in \mathbb{B}^n \qquad f(z) = \langle f, K_z \rangle.$$

Orthogonal Bergman projection of $L^2(\mathbb{B}^n)$ over $\mathcal{A}^2(\mathbb{B}^n)$:

$$\forall z \in \mathbb{B}^n \qquad \forall f \in L^2(\mathbb{B}^n) \qquad (Pf)(z) = \langle f, K_z \rangle.$$

Basic projections in $\mathcal{A}^2(\mathbb{B}^n)$

For every $\alpha \in \mathbb{N}^n$, denote by P_α the orthogonal projection over e_α:

$$P_\alpha \colon \mathcal{A}^2(\mathbb{B}^n) \to \mathcal{A}^2(\mathbb{B}^n),$$

$$P_\alpha f := \langle f, e_\alpha \rangle \, e_\alpha.$$

Since the family $(e_\alpha)_{\alpha \in \mathbb{N}^n}$ is an orthonormal basis of $\mathcal{A}^2(\mathbb{B}^n)$,
every function $f \in \mathcal{A}^2(\mathbb{B}^n)$ can be expanded
into the following serie converging in the norm $\|\cdot\|_2$:

$$f = \sum_{\alpha \in \mathbb{N}^n} P_\alpha f.$$

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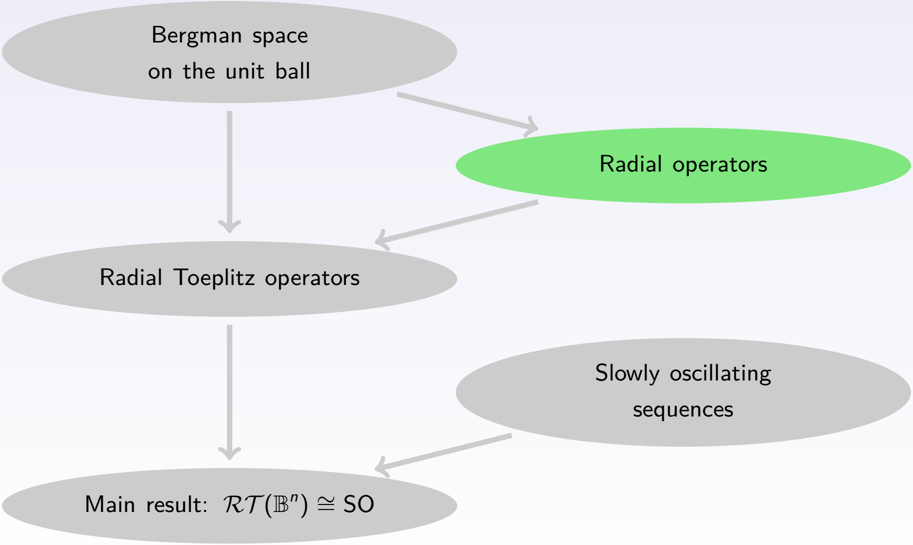
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Unitary group

The unitary group $\mathcal{U}_n$ of degree n:

$$\mathcal{U}_n := \left\{ U \in \mathbb{C}^{n\times n} : \; U^* U = I_n \right\}.$$

Examples of unitary matrices:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\,\varphi} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{U}_4, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & 0 & -\sin\varphi \\ 0 & 0 & 1 & 0 \\ 0 & \sin\varphi & 0 & \cos\varphi \end{bmatrix} \in \mathcal{U}_4.$$

Radial operators in $\mathcal{A}^2(\mathbb{B}^n)$

The unitary group $\mathcal{U}_n$ acts on the Bergman space $\mathcal{A}^2$ as follows: for every unitary matrix $U \in \mathcal{U}_n$ define

$$\Psi_U \colon \mathcal{A}^2 \to \mathcal{A}^2,$$

$$(\Psi_U f)(z) := f(U^* z).$$

Note that Ψ_U is a unitary operator.

A bounded operator $S \colon \mathcal{A}^2(\mathbb{B}^n) \to \mathcal{A}^2(\mathbb{B}^n)$ is called radial if it commutes with Ψ_U for every unitary matrix U:

$$S \text{ is radial} \quad \overset{\text{def}}{\iff} \quad \forall U \in \mathcal{U}_n \quad S\Psi_U = \Psi_U S.$$

Radial operator asociated to a bounded sequence

Given a bounded sequence $\lambda = (\lambda_j)_{j\in\mathbb{N}}$,
define the operator $R_\lambda \colon\ \mathcal{A}^2(\mathbb{B}^n) \to \mathcal{A}^2(\mathbb{B}^n)$ by the rule:

$$R_\lambda f := \sum_{\alpha\in\mathbb{N}^n} \lambda_{|\alpha|} P_\alpha f = \sum_{j=0}^{\infty} \lambda_j \left(\sum_{|\alpha|=j} P_\alpha f \right).$$

For example, if $n = 2$, then

$$\begin{aligned} R_\lambda f &= \lambda_0 P_{(0,0)} f \\ &\quad + \lambda_1 P_{(1,0)} f + \lambda_1 P_{(0,1)} f \\ &\quad + \lambda_2 P_{(2,0)} f + \lambda_2 P_{(1,1)} f + \lambda_2 P_{(0,2)} f \\ &\quad + \ldots . \end{aligned}$$

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If $n = 3$, then

$$\begin{aligned} R_\lambda f &= \lambda_0 P_{(0,0,0)} f \\ &\quad + \lambda_1 \left(P_{(1,0,0)} + P_{(0,1,0)} + P_{(0,0,1)} \right) f \\ &\quad + \lambda_2 \left(P_{(2,0,0)} + P_{(1,1,0)} + P_{(1,0,1)} + P_{(0,2,0)} + P_{(0,1,1)} + P_{(0,0,2)} \right) f \\ &\quad + \dots . \end{aligned}$$

Note that

- R_λ is diagonal with respect to the canonical basis,
- the eigenvalue associated to e_α depends only on $|\alpha|$.

It can be shown that R_λ is a radial operator.

Criterion for an operator in $\mathcal{A}^2(\mathbb{B}^n)$ to be radial

Theorem

Let $S\colon \mathcal{A}^2(\mathbb{B}^n) \to \mathcal{A}^2(\mathbb{B}^n)$ be a bounded linear operator. Then

$$S \text{ is radial} \quad \Longleftrightarrow \quad \exists \lambda \in \ell^\infty \quad S = R_\lambda.$$

In other words, a bounded linear operator S is radial
$\Longleftrightarrow$ it satisfies two conditions:

- it is diagonal with respect to the canonical basis $(e_\alpha)_{\alpha\in\mathbb{N}^n}$;
- the eigenvalue of S associated to e_α depends only on $|\alpha|$.

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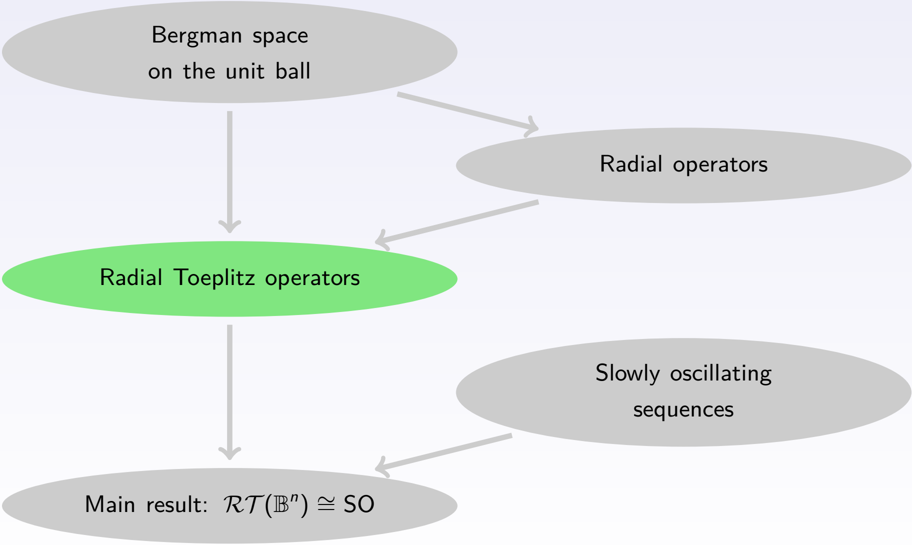
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Toeplitz operators on $\mathcal{A}^2(\mathbb{B}^n)$

Given a function $g \in L^\infty(\mathbb{B}^n)$,
the Toeplitz operator $T_g \colon \mathcal{A}^2(\mathbb{B}^n) \to \mathcal{A}^2(\mathbb{B}^n)$ is defined by:

$$T_g f := P(gf).$$

In this work we restrict ourself to radial Toeplitz operators.

Criterion of radial Toeplitz operators

Ze-Hua Zhou, Wei-Li Chen, Xing-Tang Dong (2011).
The Berezin transform and radial operators
on the Bergman space of the unit ball.
Complex Analysis and Operator Theory.
http://dx.doi.org/10.1007/s11785-011-0145-2

Theorem (Zhou, Chen, Dong 2011)

Let $g \in L^\infty(\mathbb{B}^n)$. Then

$$T_g \text{ is radial} \qquad \Longleftrightarrow \qquad g \text{ is radial.}$$

The condition "g is radial" means that
there exists a function $a \in L^\infty[0,1]$ such that

$$g(z) = a(\|z\|) \qquad \text{a.e.} \quad z \in \mathbb{B}^n.$$

Sequence of eigenvalues of a radial Toeplitz operators

Sergei Grudsky, Alexei Karapetyants, Nikolai Vasilevski (2003).
Toeplitz operators on the unit ball in $\mathbb{C}^n$ with radial symbols.
Journal of Operator Theory.

Theorem (Grudsky, Karapetyants, Vasilevski 2003)

Let $a \in L^\infty[0,1]$. Denote by $\tilde{a}$ the radial extention of a to the unit ball:

$$\tilde{a}\colon \mathbb{B}^n \to \mathbb{C}, \qquad \tilde{a}(z) := a(\|z\|).$$

Then the operator $T_{\tilde{a}}$ is radial: $T_{\tilde{a}} = R_{\gamma_{n,a}}$,
and the sequence $\gamma_{n,a}$ of its eigenvalues can be computed by the formula:

$$\gamma_{n,a}(j) = (n+j)\int_0^1 a(\sqrt{r}\,)\, r^{n+j-1}\, dr.$$

Passing from operators to sequences

$T_{\tilde{a}}$ $\gamma_{n,a}$

$\Xi_n := \{ T_{\tilde{a}} : \ a \in L^\infty[0,1]\}$ $\Gamma_n := \{\gamma_{n,a} : \ a \in L^\infty[0,1]\}$

$\mathcal{RT}(\mathbb{B}^n) := C^*\text{-algebra}(\Xi_n)$ $\mathcal{A}_n := C^*\text{-algebra}(\Gamma_n)$

The mapping $T_{\tilde{a}} \mapsto \gamma_{n,a}$ is linear, multiplicative and isometric.

The C^*-algebras $\mathcal{RT}(\mathbb{B}^n)$ and $\mathcal{A}_n$ are isometrically isomorphic.

Sequences of eigenvalues of radial Toeplitz operators

$$\gamma_{n,a}(j) = (n+j) \int_0^1 a(\sqrt{r}\,)\, r^{n+j-1}\, dr \in \ell^\infty.$$

$$\Gamma_n := \{\gamma_{n,a} : \ a \in L^\infty[0,1]\} \subset \ell^\infty.$$

$$\mathcal{A}_n := C^*\text{-algebra}(\Gamma_n) \subset \ell^\infty.$$

The set Γ_n can be described in terms of iterated differences, but this description is rather complicated.

Problems:

(A) Describe the closure of Γ_n in ℓ^∞.

(B) Describe the C^*-algebra $\mathcal{A}_n$ generated by Γ_n.

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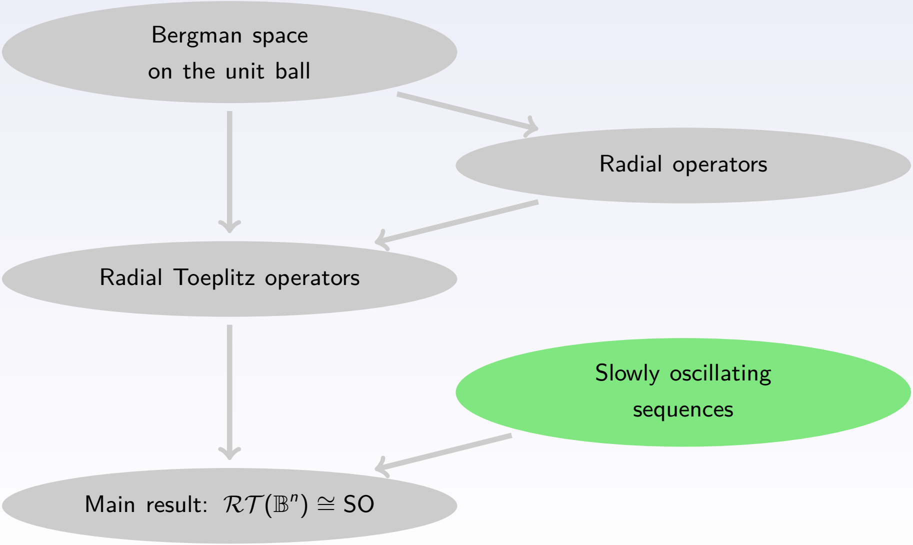
It results that the problems (A) and (B) have the same answer:

slowly oscillating sequences.

Outline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Slowly oscillating sequences

Were introduced by Robert Schmidt in 1925:

Robert Schmidt (1925):
Über divergente Folgen and lineare Mittelbildungen,
Mathematische Zeitschrift.
http://dx.doi.org/10.1007/BF01479600

$$\mathrm{SO} := \left\{ \lambda \in \ell^\infty : \quad \lim_{\frac{j+1}{k+1} \to 1} \left| \lambda_j - \lambda_k \right| = 0 \right\}.$$

Slowly oscillating sequences have many applications in Tauberian theory.

Formal definition of slowly oscillating sequences

"Logarithmic metric" in $\mathbb{N}$:

$$\rho(j,k) := \left|\log(j+1) - \log(k+1)\right| = \log\frac{\max\{j+1,k+1\}}{\min\{j+1,k+1\}}.$$

Modulus of continuity of a sequence $\lambda = (\lambda_j)_{j\in\mathbb{N}}$ with respect to ρ:

$$\omega_{\rho,\lambda}\colon\ [0,+\infty) \to [0,+\infty],$$

$$\omega_{\rho,\lambda}(\delta) := \sup\left\{|\lambda_j - \lambda_k|\colon\ \rho(j,k) \le \delta\right\}.$$

Slowly oscillating sequences

are bounded uniformly continuous functions $(\mathbb{N},\rho) \to \mathbb{C}$:

$$\mathrm{SO} := \left\{\lambda \in \ell^\infty\colon\ \lim_{\delta\to 0^+} \omega_{\rho,\lambda}(\delta) = 0\right\}.$$

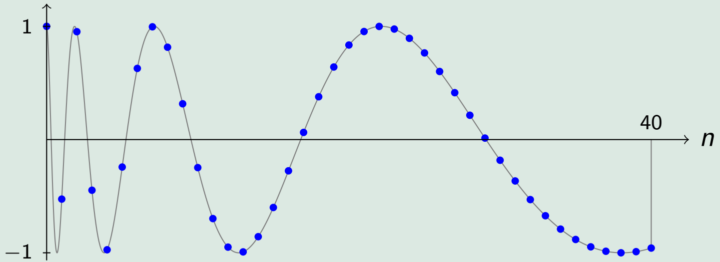
Examples of sequences in SO

Every convergent sequence belongs to SO:

$$c \subset \mathrm{SO}\,.$$

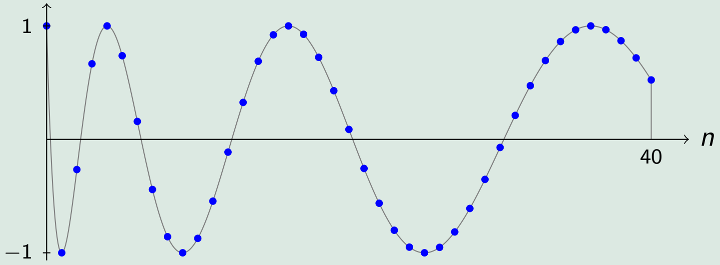
Example of a sequence $x \in \mathrm{SO} \setminus c$

$$x_n = \cos\Big(6\log(n+1)\Big)$$



Example of a sequence that does not belong to SO

$$x_j = \cos(\pi\sqrt{j}\,)$$



This sequence satisfies the property

$$|x_{j+1} - x_j| \to 0 \qquad \text{as} \qquad j \to \infty,$$

but does not belong to the class SO of Schmidt:

$$|x_j - x_k| \not\to 0 \qquad \text{as} \qquad \frac{j+1}{k+1} \to 1.$$

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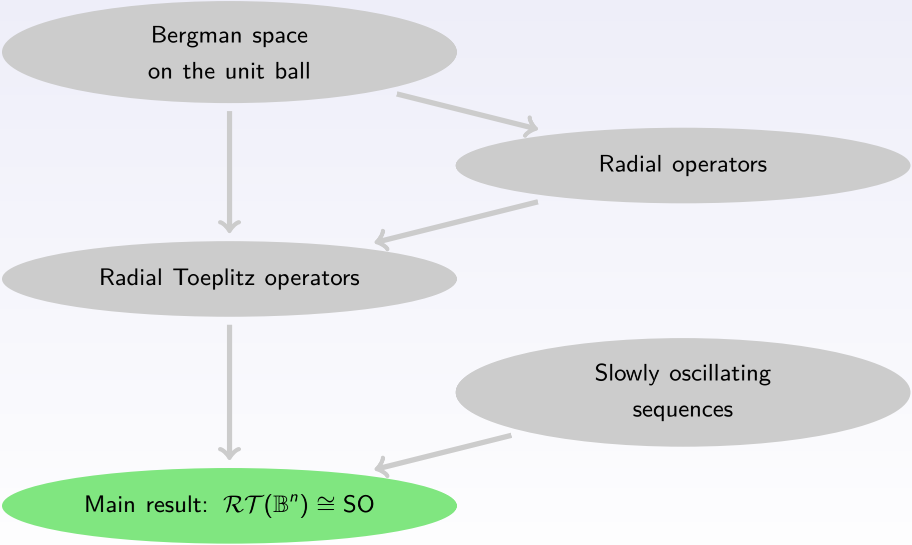
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Approximation tecnics by Daniel Suárez

Daniel Suárez (2008):
The eigenvalues of limits of radial Toeplitz operators.
Bulletin of the London Mathematical Society.
http://dx.doi.org/10.1112/blms/bdn042

(For the one-dimensional case)

Theorem (Suárez, 2008)

The C^*-algebra $\mathcal{A}_1$ generated by Γ_1 coincides with the closure of Γ_1 and is equal to the closure of the class d_1, where

$$d_1 := \left\{ y \in \ell^\infty : \quad \sup_{j \geq 0} \Big((j+1) \left| y_{j+1} - y_j \right| \Big) < +\infty \right\}.$$

The proof of Suárez is based on the so-called m-Berezin transform.

Main result

Recall that $\mathcal{A}_n$ is the C^*-algebra generated by $\Gamma_n = \{\gamma_{n,a} : \; a \in L^\infty[0,1]\}$.

Theorem

$$\mathcal{A}_n \quad = \quad \textit{the closure of } \Gamma_n \textit{ in } \ell^\infty \quad = \quad \mathrm{SO}\,.$$

As a corollary,

$$\mathcal{RT}(\mathbb{B}^n) \cong \mathrm{SO}\,.$$

Idea of the proof.

$$\mathcal{RT}(\mathbb{B}^n) \cong \mathcal{A}_n = \mathcal{A}_1 = \overline{d_1} = \mathrm{SO}\,.$$

□

Example

Define a sequence $\lambda = (\lambda_j)_{j \in \mathbb{N}}$ by

$$\lambda_j := \exp\left(\frac{\mathrm{i}}{3\pi} \ln^2(j+n)\right).$$

Then $\lambda \in \ell^\infty(\mathbb{N}) \setminus \mathrm{SO}(\mathbb{N})$ and there exists a function $a \in L^1([0,1], r\,dr)$ such that $\lambda = \gamma_{n,a}$.

So, the corresponding radial Toeplitz opeator $T_{\tilde{a}}$ is bounded, but does not belong to $\mathcal{RT}(\mathbb{B}^n)$.