Radial Toeplitz operators on the Bergman space and very slowly oscillating sequences

Egor Maximenko

joint work with
Nikolai Vasilevski and Sergey Grudsky

Instituto Politécnico Nacional, ESFM, México

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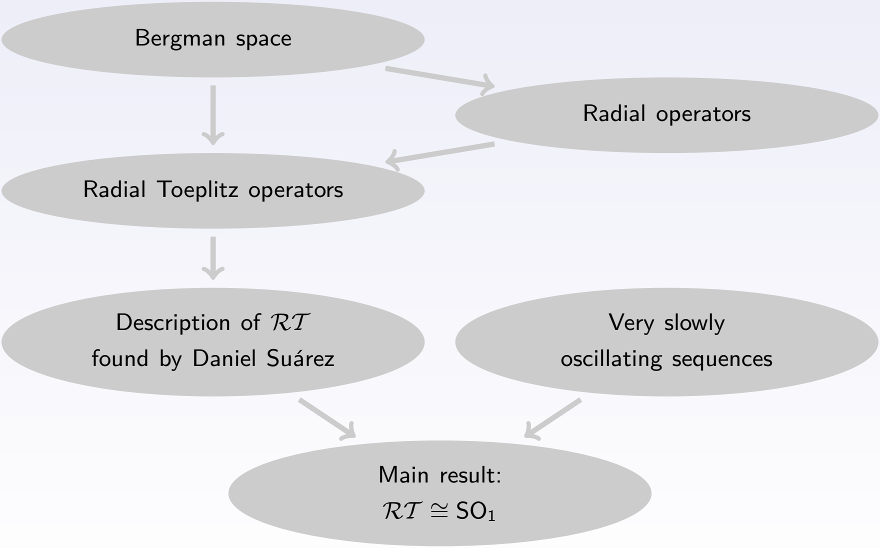
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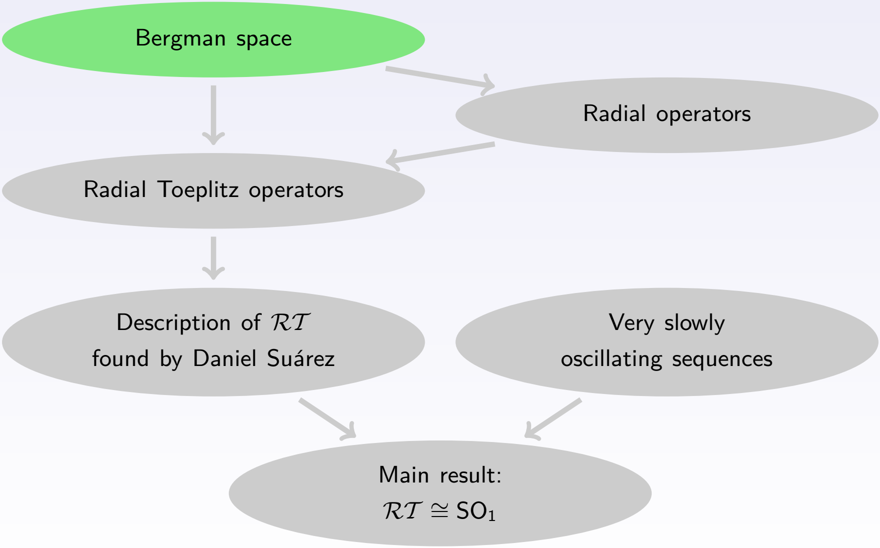
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Bergman space

$\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$.

$\mathbb{D}$ is considered with the Lebesgue plane measure μ.

$A^2(\mathbb{D}) =$ Bergman space $:= \{f \in L^2(\mathbb{D}) \colon \; f \text{ is analytic}\}$.

$A^2(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D})$.

Monomial basis and Bergman projection

The normalized monomials form an orthonormal basis of $A^2(\mathbb{D})$:

$$\varphi_n(z) := \sqrt{\tfrac{n+1}{\pi}}\, z^n.$$

Bergman kernel:

$$K(z,w) = \sum_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_n(w)} = \frac{1}{\pi(1 - z\overline{w})^2}.$$

Bergman projection:

$$(Bf)(z) := \sum_{n=0}^{\infty} \langle f, \varphi_n \rangle\, \varphi_n = \int_{\mathbb{D}} K(z,w)\, f(w)\, d\mu(w).$$

B is an orthonormal projection of $L^2(\mathbb{D})$ onto $A^2(\mathbb{D})$.

Isometric isomorphism between $A^2(\mathbb{D})$ and ℓ^2

Since $A^2(\mathbb{D})$ is an infinite-dimensional separable Hilbert space,

$$A^2(\mathbb{D}) \cong \ell^2,$$

where $\ell^2 :=$ the space of the quadratically summable complex sequences.

Using the monomial basis $(\varphi_n)_{n=0}^{\infty}$ of $A^2(\mathbb{D})$
define the following isometric isomorphism

$$U\colon A^2(\mathbb{D}) \to \ell^2.$$

$$\begin{aligned} U\colon \quad & f \in A^2(\mathbb{D}) & \mapsto \quad & (\langle f, \varphi_n \rangle)_{n=0}^{\infty} \in \ell^2. \\ U^{-1}\colon \quad & (x_n)_{n=0}^{\infty} \in \ell^2 & \mapsto \quad & \sum_{n=0}^{\infty} x_n \varphi_n \in A^2(\mathbb{D}). \end{aligned}$$

Evaluation functionals and Berezin transform

$$\forall f \in A^2(\mathbb{D}) \qquad \forall z \in \mathbb{D} \qquad f(z) = \langle f, K_z \rangle,$$

where

$$K_z(\xi) = \overline{K(z,\xi)} = \frac{1}{\pi(1-\overline{z}\xi)^2}.$$

Denote by k_z the function K_z after the normalization:

$$k_z := \frac{K_z}{\|K_z\|}, \qquad k_z(\xi) = \frac{1-|z|^2}{\sqrt{\pi}(1-\overline{z}\xi)^2}.$$

The Berezin transform $\widetilde{S}$ of a bounded linear operator $S\colon A^2(\mathbb{D}) \to A^2(\mathbb{D})$ is defined by:

$$\widetilde{S}\colon \mathbb{D} \to \mathbb{C}, \qquad \widetilde{S}(z) := \langle S k_z, k_z \rangle.$$

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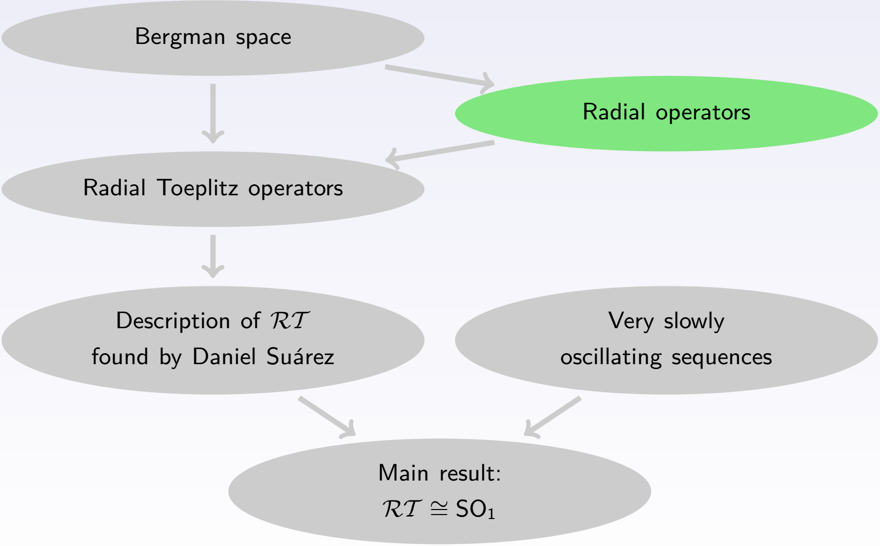

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Rotations of the unit disk

Given a real number θ, consider the operator $R_\theta\colon A^2(\mathbb{D}) \to A^2(\mathbb{D})$ which "rotates the functions through the angle θ":

$$(R_\theta f)(z) := f(e^{-i\theta} z).$$

R_θ is an isometrical isomorphism, and its inverse is $R_{-\theta}$:

$$R_\theta^{-1} = R_\theta^* = R_{-\theta}.$$

The monomials are eigenvectors of R_θ:

$$R_\theta \varphi_n = e^{-ni\theta} \varphi_n.$$

R_θ interacts very well with the normalized reproducing kernel:

$$R_\theta k_z = k_{e^{i\theta} z}.$$

A function $f\colon \mathbb{D} \to \mathbb{C}$ is called radial if

$$\forall z \in \mathbb{D} \qquad f(z) = f(|z|).$$

Criterion of radial operators (N. Zorboska, 2002)

Let $S\colon A^2(\mathbb{D}) \to A^2(\mathbb{D})$ be a bounded linear operator.
Then the following conditions are equivalent:

(a) S is invariant under rotations:

$$\forall \theta \in \mathbb{R} \qquad R_\theta S = R_\theta S.$$

(b) there exists a sequence $b \in \ell^\infty$ such that

$$\forall n \in \{0, 1, 2, \ldots\} \qquad S\varphi_n = b_n \varphi_n;$$

in other words,

$$S = U^{-1} M_b U,$$

where M_b is the multiplication by the sequence b;

(c) the Berezin transform of S is radial:

$$\forall z \in \mathbb{D} \qquad \widetilde{S}(z) = \widetilde{S}(|z|).$$

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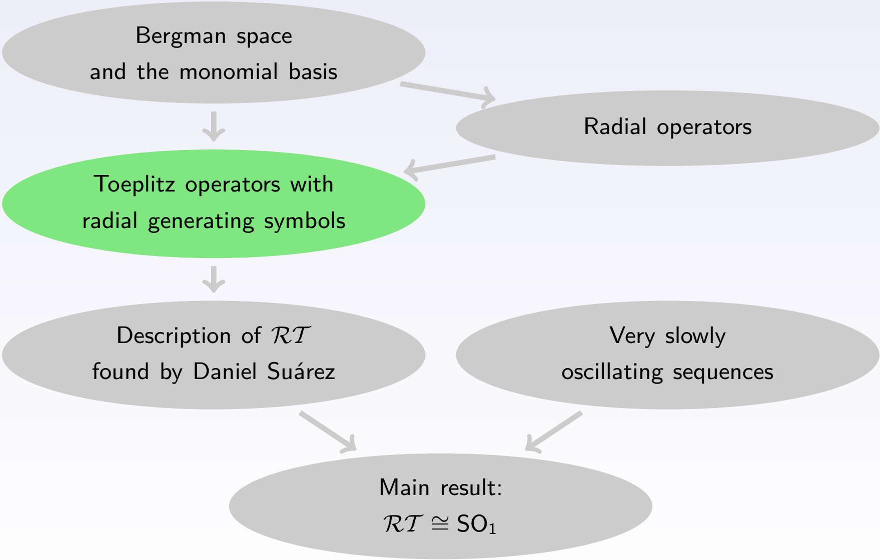
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Radial Toeplitz operators on the Bergman space

Given a function $a \in L^\infty(\mathbb{D})$,
the Toeplitz operator with defining symbol a is

$$T_a \colon A^2(\mathbb{D}) \to A^2(\mathbb{D}), \qquad T_a f = B(af).$$

Criterion of radial Toeplitz operator

$$T_a \text{ is radial} \quad \Longleftrightarrow \quad a \text{ is radial.}$$

Here we consider only radial Toeplitz operators,
i.e. the Toeplitz operators with radial defining symbols.

We identify a radial function with its restriction to the interval $[0, 1)$.

Algebra generated by radial Toeplitz operators

Consider the set of all Toeplitz operators
with bounded radial defining symbols:

$$\Lambda := \{ T_a : \; a \in L^\infty(0,1) \}.$$

The object of this work is the C^*-algebra generated by these operators:

$$\mathcal{RT} := C^*\text{-algebra generated by } \Lambda.$$

Diagonalization of the radial Toeplitz operators (Korenblum and Zhu, 1995)

Theorem

Let $a \in L^\infty(\mathbb{D})$ be a radial function. Then T_a is diagonal with respect to the monomial basis:

$$\forall n \in \{0, 1, 2, \ldots\} \qquad T_a \varphi_n = \gamma_a(n) \varphi_n,$$

where the sequence γ_a of the corresponding eigenvalues is defined by

$$\gamma_a(n) = (n+1) \int_0^1 a(\sqrt{r}) r^n \, dr.$$

In other words,

$$U T_a U^{-1} = M_{\gamma_a}.$$

Passing from
operators to sequences

T_a γ_a

$\Lambda := \{ T_a \colon a \in L^\infty(0,1)\}$ $\Gamma := \{\gamma_a \colon a \in L^\infty(0,1)\}$

$\mathcal{RT} := C^*\text{-algebra}(\Lambda)$ $\mathcal{A} := C^*\text{-algebra}(\Gamma)$

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T_a	γ_a
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The mappings $T_a \mapsto M_{\gamma_a} \mapsto \gamma_a$ are linear, multiplicative and isometric. Therefore,

The C^*-algebras $\mathcal{RT}$ and $\mathcal{A}$ are isometrically isomorphic.

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The aim of this work: describe $\mathcal{A}$ in the explicit manner.

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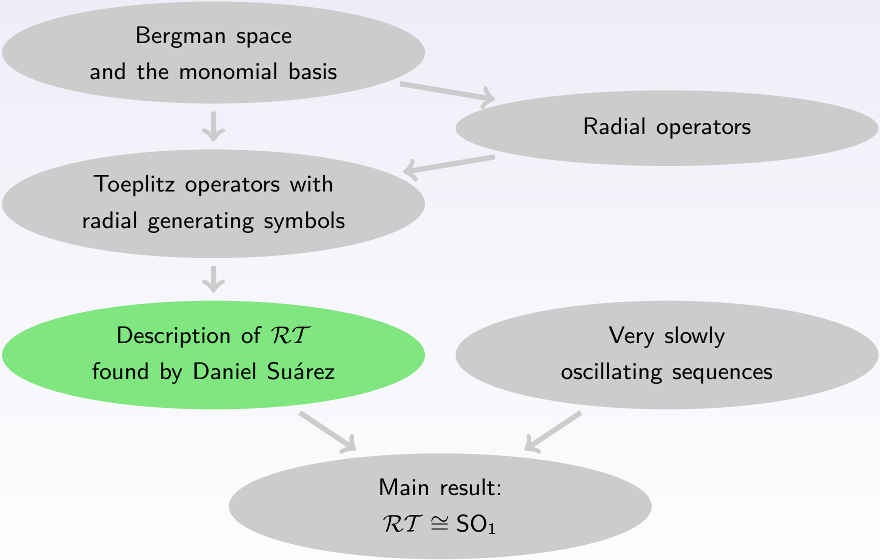
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Iterated differences of a sequence

Given a sequence $x = (x_n)_{n=0}^{\infty}$, define its differences of the first order:

$$(\Delta x)_n = x_{n+1} - x_n,$$

the differences of the second order:

$$(\Delta^2 x)_n = (\Delta x)_{n+1} - (\Delta x)_n = x_{n+2} - 2x_{n+1} + x_n,$$

etc.

Differences of the order k:

$$(\Delta^k x)_n = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} x_{n+j}.$$

A classical result from the moment theory

Given a function $b\colon [0,1] \to \mathbb{C}$, its moment sequence is $(\mu_b(n))_{n=0}^{\infty}$,

$$\mu_b(n) := \int_0^1 b(t) t^n \, dt.$$

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Criterion of moment sequence of a bounded function (based on results of Hausdorff)

Given a sequence $(x_n)_{n=0}^{\infty}$, the following conditions are equivalent:

- $(x_n)_{n=0}^{\infty}$ is the moment sequence of a bounded function;
- $\sup\limits_{k,n\geq 0} \left| (n+k+1)\binom{n+k}{k}(\Delta^k x)_n \right| < +\infty.$

Description of Γ through iterated differences

Recall that $\Gamma := \{\gamma_a : \; a \in L^\infty(0,1)\}$, where

$$\gamma_a(n) = (n+1)\int_0^1 a(\sqrt{r})r^n\,dr.$$

Compare to the definition of the moment sequence:

$$\mu_b(n) := \int_0^1 b(t)t^n\,dt.$$

Description of Γ through iterated differences (Daniel Suárez, 2008)

A sequence $x = (x_n)_{n=0}^\infty$ belongs to Γ $\iff$

$$\sup_{k,n\geq 0}\left|(n+k+1)\binom{n+k}{k}(\Delta^k y)_n\right| < +\infty \qquad \text{where} \quad y_n = \frac{x_n}{n+1}.$$

Description of $\mathcal{A}$ by Suárez

Theorem (Daniel Suárez, 2005)

The set Γ not only generates the C^*-algebra $\mathcal{A}$, but is also dense in $\mathcal{A}$:

$$\mathcal{A} \quad \text{is the closure of} \quad \Gamma \quad \text{in} \quad \ell^\infty.$$

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Let d_1 be the set of all bounded sequences $(x_n)_{n=0}^\infty$ such that

$$\sup_{n\geq 0}\Big((n+1)\,|x_{n+1}-x_n|\Big) < +\infty.$$

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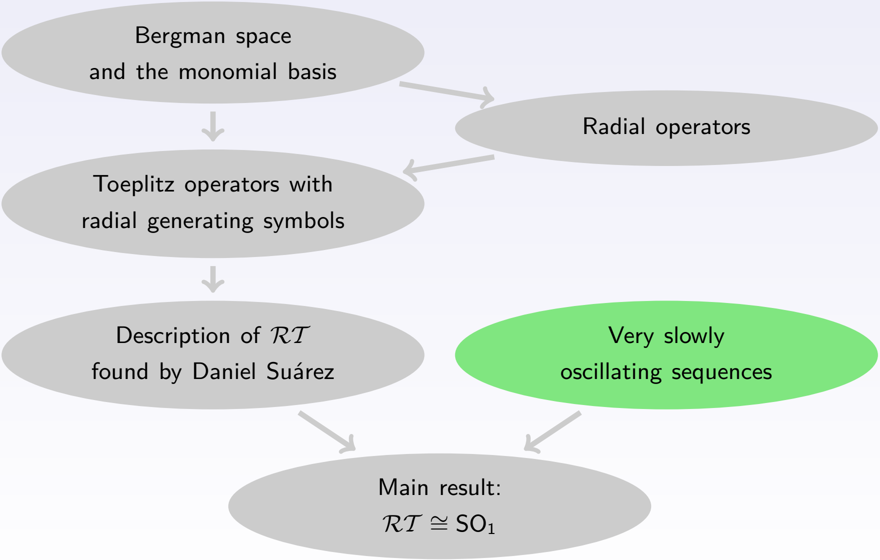
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The algebra SO_1

SO_1 := the set of all bounded sequences $x = (x_n)_{n=0}^{\infty}$ such that

$$\lim_{\frac{m}{n}\to 1} |x_m - x_n| = 0,$$

that is,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall m, n \qquad \left(\left| \frac{m}{n} - 1 \right| < \delta \quad \Rightarrow \quad |x_m - x_n| < \varepsilon \right).$$

Observation

SO_1 is a C^*-subalgebra of ℓ^∞.

Examples of sequences in SO_1

Sequences that have a finite limit

$$c \subsetneq \mathrm{SO}_1$$

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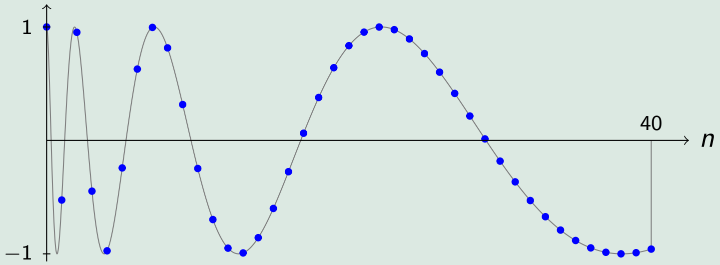
Example of a sequence $x \in \mathrm{SO}_1$ such that $\nexists \lim\limits_{n\to\infty} x_n$

$$x_n = \cos\Big(6\log(n+1)\Big)$$



Comparison with the algebra SO

SO consists of the bounded sequences $x = (x_n)_{n=0}^{\infty}$ such that

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Proposition

$$\mathrm{SO}_1 \subsetneq \mathrm{SO}.$$

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Proposition

$$\mathrm{SO}_1 \subsetneq \mathrm{SO}.$$

Idea of the proof

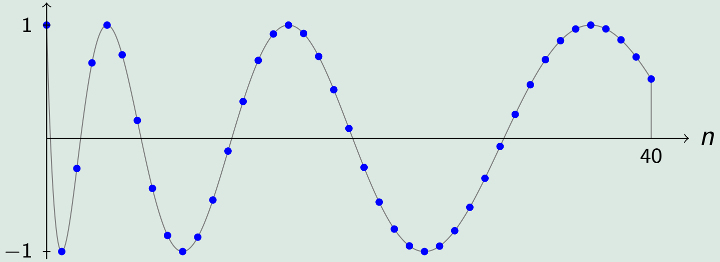
If $n \to \infty$ and $m = n + 1$, then $\frac{m}{n} \to 1$. Therefore $\mathrm{SO}_1 \subseteq \mathrm{SO}$.

To show that $\mathrm{SO}_1 \neq \mathrm{SO}$, consider the sequence

$$x_n = \cos(\pi\sqrt{n}).$$

Example of a sequence $\in \mathrm{SO} \setminus \mathrm{SO}_1$

$$x_n = \cos\left(\pi\sqrt{n}\right)$$



This sequence belongs to SO:

$$|x_{n+1} - x_n| \to 0 \qquad \text{as} \qquad n \to \infty,$$

but doesn't belong to SO_1:

$$|x_m - x_n| \not\to 0 \qquad \text{as} \qquad \frac{m}{n} \to 1.$$

d_1 is a dense proper subset of SO_1

Recall that d_1 consists of the bounded sequences $(y_n)_{n=0}^{\infty}$ such that

$$\sup_{n\geq 0}\Big((n+1)\left|y_{n+1}-y_n\right|\Big) < +\infty.$$

Sketch of the proof that d_1 is dense in SO_1.

Given $x \in SO_1$ and $\delta > 0$ construct the sequence $y^{(\delta)}$ using the following averaging technique:

$$y_n^{(\delta)} := \frac{1}{\lceil n\delta \rceil} \sum_{k=n}^{n+\lceil \delta n \rceil} x_k.$$

Then $y^{(\delta)} \in d_1$.

The condition $x \in SO_1$ implies that $\|y^{(\delta)} - x\|_\infty \to 0$ as $\delta \to 0$. □

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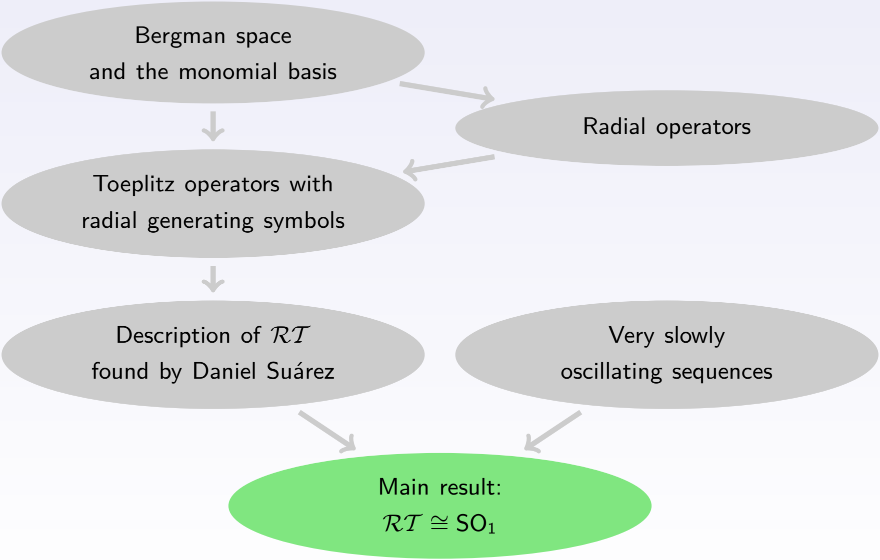
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Main result

Theorem

$$\mathcal{A} = \mathsf{SO}_1 \,.$$

As a corollary,

$$\mathcal{RT} \cong \mathsf{SO}_1 \,.$$

Proof.

The result of Suárez says that d_1 is a dense subset of $\mathcal{A}$.

We have proven that d_1 is a dense subset of SO_1. So,

$$\mathcal{RT} \cong \mathcal{A} = \overline{d_1} = \mathsf{SO}_1 \,.$$

□