Radial Toeplitz operators on the Bergman space and slowly oscillating sequences

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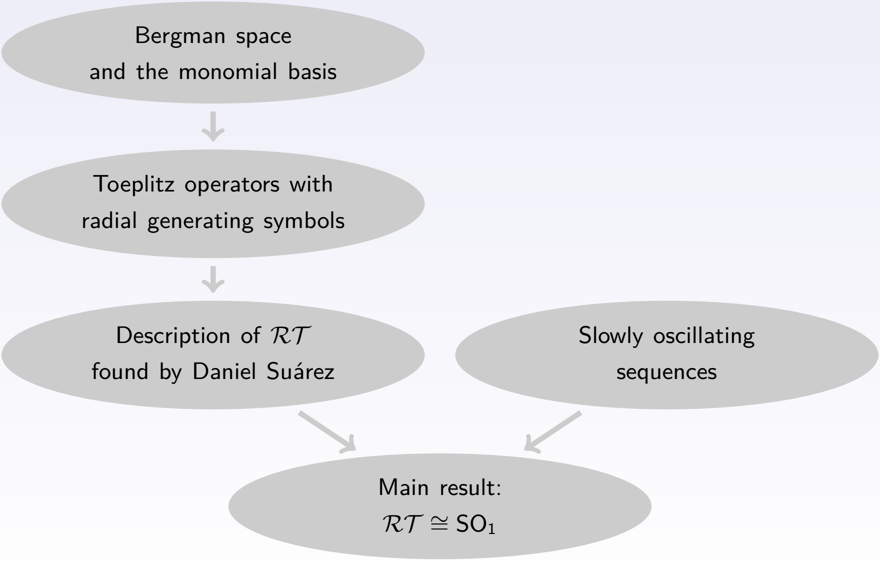
The results are joint with

Nikolai Vasilevski and Sergey Grudsky.

Contents


Bergman space
and the monomial basis
Toeplitz operators with
radial generating symbols
Description of $\mathcal{RT}$
found by Daniel Suárez
Slowly oscillating
sequences
Main result:
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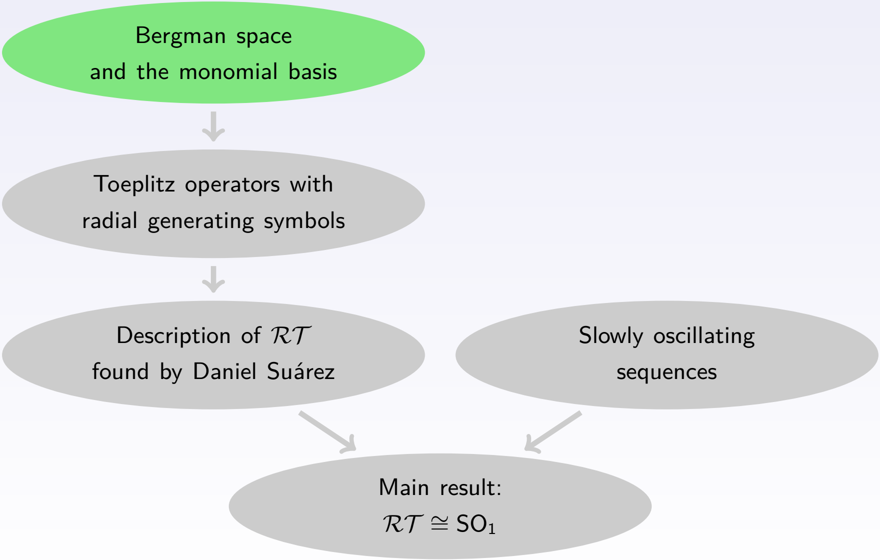
Contents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and the monomial basis
Toeplitz operators with
radial generating symbols
Description of $\mathcal{RT}$
found by Daniel Suárez
Slowly oscillating
sequences
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Bergman space

$\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$.

$\mu :=$ plane Lebesgue measure.

$A^2(\mathbb{D}) =$ Bergman space $:= \{f \in L^2(\mathbb{D}, \mu) \colon \ f$ is analitic$\}$.

$A^2(\mathbb{D})$ is a (closed) subspace of $L^2(\mathbb{D}, \mu)$.

Monomial basis, Bergman projection

The normalized monomials form an orthonormal basis of $A^2(\mathbb{D})$:

$$\varphi_n(z) := \sqrt{\tfrac{n+1}{\pi}}\, z^n.$$

Bergman kernel:

$$K(z,w) = \sum_{n=0}^{\infty} \varphi_n(z)\overline{\varphi_n(w)} = \frac{1}{\pi(1 - z\overline{w})^2}.$$

Bergman projection:

$$(Bf)(z) := \sum_{n=0}^{\infty} \langle f, \varphi_n \rangle\, \varphi_n = \int_{\mathbb{D}} K(z,w)\, f(w)\, d\mu(w).$$

B is an orthonormal projection of $L^2(\mathbb{D},\mu)$ onto $A^2(\mathbb{D})$.

Isometric isomorphism between $A^2(\mathbb{D})$ and ℓ^2

Since $A^2(\mathbb{D})$ is an infinite-dimensional separable Hilbert space,

$$A^2(\mathbb{D}) \cong \ell^2,$$

where $\ell^2 :=$ the space of quadratically summable complex sequences.

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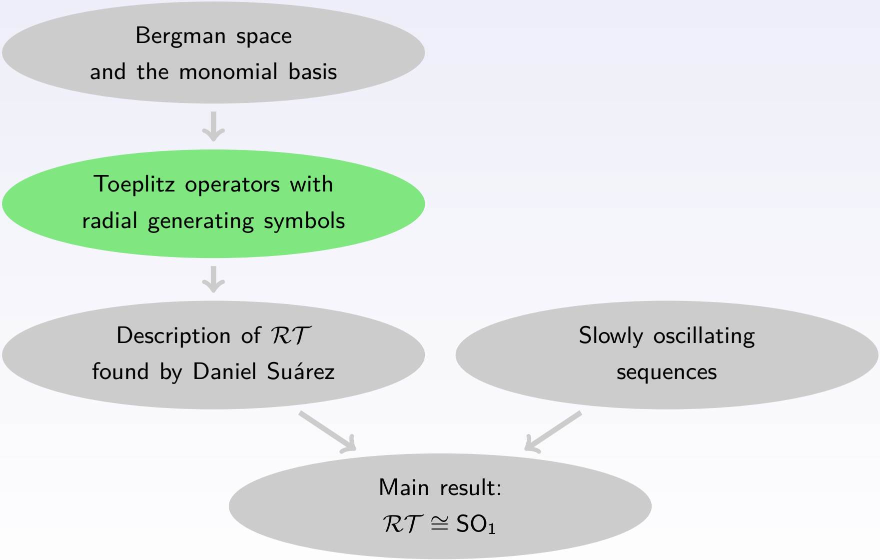
Using the monomial basis $(\varphi_n)_{n=0}^{\infty}$ of $A^2(\mathbb{D})$
define the following isometric isomorphism

$$R\colon A^2(\mathbb{D}) \to \ell^2.$$

$$\begin{array}{llll} R\colon & f \in A^2(\mathbb{D}) & \mapsto & (\langle f, \varphi_n \rangle)_{n=0}^{\infty} \in \ell^2. \\ R^{-1}\colon & (x_n)_{n=0}^{\infty} \in \ell^2 & \mapsto & \sum_{n=0}^{\infty} x_n \varphi_n \in A^2(\mathbb{D}). \end{array}$$

Contents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and the monomial basis
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Radial Toeplitz operators on the Bergman space

Given a function $a \in L^\infty(\mathbb{D})$,
the Toeplitz operator with defining symbol a is

$$T_a \colon A^2(\mathbb{D}) \to A^2(\mathbb{D}), \qquad T_a f = B(af).$$

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i.e. the Toeplitz operators with radial defining symbols.

A function $a \colon \mathbb{D} \to \mathbb{C}$ is called radial if

$$a(z) = a(|z|) \qquad \forall z \in \mathbb{D}.$$

We identify a radial function with its restriction to the interval $[0, 1)$.

Algebra generated by radial Toeplitz operators

Consider the set of all Toeplitz operators
with bounded radial defining symbols:

$$\Lambda := \{ T_a : \ a \in L^\infty(0,1) \}.$$

The object of this work is the C^*-algebra generated by these operators:

$$\mathcal{RT} := C^*\text{-algebra generated by } \Lambda.$$

Diagonalization of radial Toeplitz operators

Theorem (Korenblum and Zhu, 1995)

Let $a \in L^\infty(\mathbb{D})$ be a radial function. Then

$$RT_aR^{-1} = M_{\gamma_a},$$

where $M_{\gamma_a}\colon \ell^2 \to \ell^2$ is the multiplication operator by the following sequence $\gamma_a \in \ell^\infty$:

$$\gamma_a(n) = (n+1)\int_0^1 a(\sqrt{r})r^n\,dr.$$

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$$\gamma_a(n) = (n+1) \int_0^1 a(\sqrt{r}) r^n \, dr.$$

In other words, the monomial basis $(\varphi_n)_{n=0}^\infty$ diagonalizes T_a, and the eigenvalues of T_a are $\gamma_a(n)$, $n = 0, 2, 3, \ldots$.

Proof of the theorem about the diagonalization

$$
\begin{aligned}
\langle T_a\varphi_n, \varphi_k\rangle &= \langle B(a\varphi_n), \varphi_k\rangle = \langle a\varphi_n, B(\varphi_k)\rangle = \langle a\varphi_n, \varphi_k\rangle \\
&= \frac{n+1}{\pi}\int_{\mathbb{D}} a(z) z^n \overline{z}^k \, d\mu(z) \\
&= \frac{n+1}{\pi}\int_0^{2\pi} e^{i(n-k)\varphi} \, d\varphi \int_0^1 a(r)\, r^{n+k+1} \, dr \\
&= \delta_{n,k} \cdot (n+1) \int_0^1 a(\sqrt{u}) u^n \, du \\
&= \delta_{n,k} \cdot \gamma_a(n).
\end{aligned}
$$

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From here

$$T_a\varphi_n = \gamma_a(n)\varphi_n$$

and

$$RT_aR^{-1}x = \gamma_a x \qquad \forall x \in \ell^2.$$

Passing from

operators to sequences

operators	sequences
T_a	γ_a
$\Lambda := \{ T_a : \; a \in L^\infty(0,1)\}$	$\Gamma := \{\gamma_a : a \in L^\infty(0,1)\}$
$\mathcal{RT} := C^*\text{-algebra}(\Lambda)$	$\mathcal{A} := C^*\text{-algebra}(\Gamma)$

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The mappings $T_a \mapsto M_{\gamma_a} \mapsto \gamma_a$ are linear, multiplicative and isometric.

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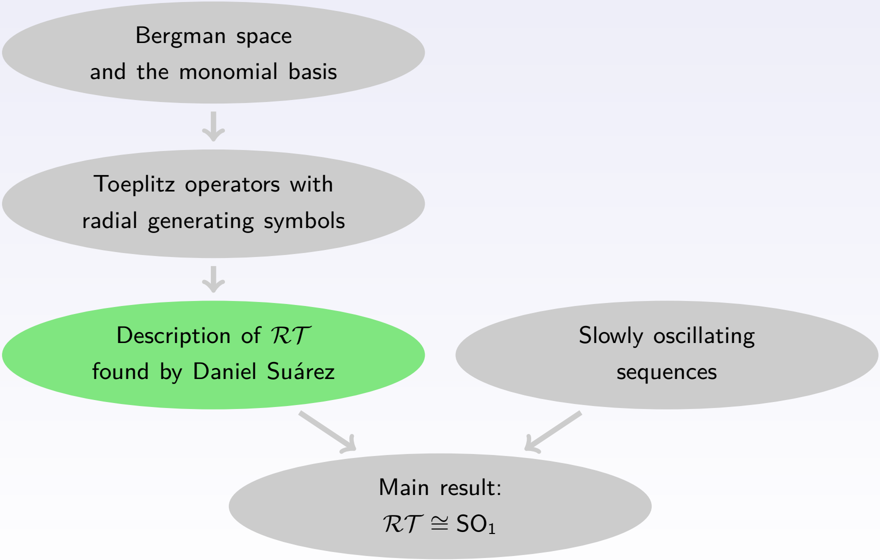
The aim of this work: describe $\mathcal{A}$ in the explicit manner.

Contents


Bergman space
and the monomial basis
Toeplitz operators with
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Iterated differences of a sequence

Given a sequence $x = (x_n)_{n=0}^{\infty}$, define its differences of the first order:

$$(\Delta x)_n = x_{n+1} - x_n,$$

the differences of the second order:

$$(\Delta^2 x)_n = (\Delta x)_{n+1} - (\Delta x)_n = x_{n+2} - 2x_{n+1} + x_n,$$

etc.

Differences of the order k:

$$(\Delta^k x)_n = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} x_{n+j}.$$

A classical result from the moment theory

Given a function $a\colon [0,1] \to \mathbb{C}$, its moment sequence is $(\mu_a(n))_{n=0}^{\infty}$,

$$\mu_a(n) := \int_0^1 a(t) t^n \, dt.$$

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Criterion of moment sequence of a bounded function (based on results of Hausdorff)

Given a sequence $(x_n)_{n=0}^{\infty}$, the following conditions are equivalent:

- $(x_n)_{n=0}^{\infty}$ is the moment sequence of a bounded function;
- $\sup_{k,n\geq 0} \left| (n+k+1) \binom{n+k}{k} (\Delta^k x)_n \right| < +\infty.$

Description of Γ through iterated differences

Recall that $\Gamma := \{\gamma_a : \ a \in L^\infty(0,1)\}$, where

$$\gamma_a(n) = (n+1)\int_0^1 a(\sqrt{r})r^n\,dr.$$

Compare to the definition of the moment sequence:

$$\mu_a(n) := \int_0^1 a(t)t^n\,dt.$$

Description of Γ through iterated differences (Daniel Suárez, 2008)

A sequence $x = (x_n)_{n=0}^\infty$ belongs to Γ $\iff$

$$\sup_{k,n\geq 0}\left|(n+k+1)\binom{n+k}{k}(\Delta^k y)_n\right| < +\infty \qquad \text{where} \quad y_n = \frac{x_n}{n+1}.$$

Description of $\mathcal{A}$ by Suárez

Theorem (Daniel Suárez, 2005)

The set Γ not only generates the C^*-algebra $\mathcal{A}$, but is also dense in $\mathcal{A}$:

$$\mathcal{A} \quad \text{is the closure of} \quad \Gamma \quad \text{in} \quad \ell^\infty.$$

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Let d_1 be the set of all bounded sequences $(x_n)_{n=0}^\infty$ such that

$$\sup_{n\geq 0}\Big((n+1)|x_{n+1}-x_n|\Big) < +\infty.$$

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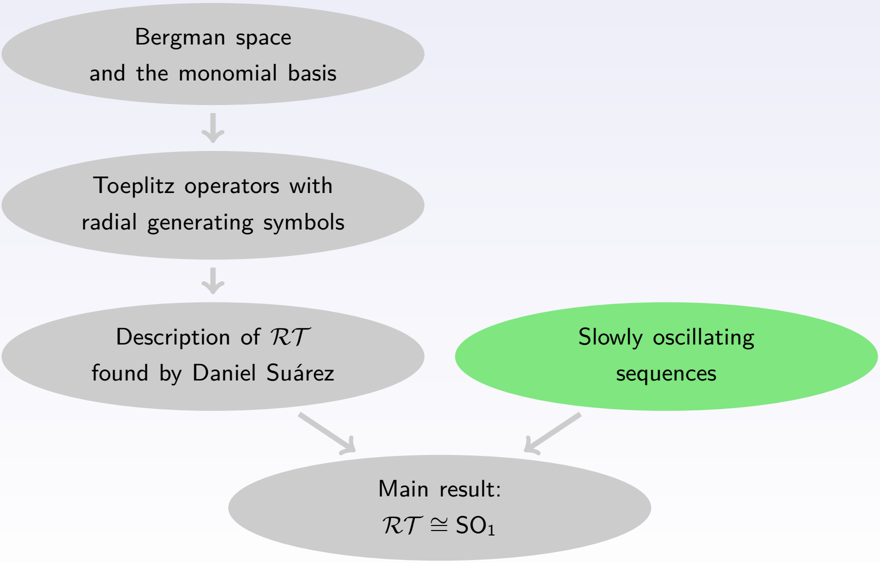
Contents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and the monomial basis
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The algebra SO_1

SO_1 := the set of all bounded sequences $x = (x_n)_{n=0}^{\infty}$ such that

$$\lim_{\frac{m}{n} \to 1} |x_m - x_n| = 0,$$

that is,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall m, n \qquad \left(\left| \frac{m}{n} - 1 \right| < \delta \quad \Rightarrow \quad |x_m - x_n| < \varepsilon \right).$$

Observation

SO_1 is a C^*-subalgebra of ℓ^∞.

Examples of sequences in SO_1

Sequences that have a finite limit

$$c \subsetneq SO_1$$

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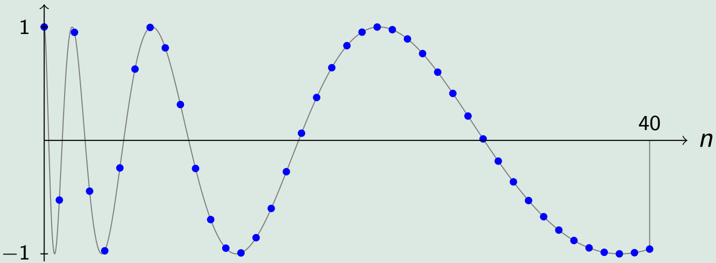
Example of a sequence $x \in \mathrm{SO}_1$ such that $\nexists \lim_{n\to\infty} x_n$

$$x_n = \cos\Big(6\log(n+1)\Big)$$



Comparison with the algebra SO

SO consists of the bounded sequences $x = (x_n)_{n=0}^{\infty}$ such that

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Proposition

$$SO_1 \subsetneq SO.$$

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$$\lim_{n\to\infty} |x_{n+1} - x_n| = 0.$$

Proposition

$$\mathrm{SO}_1 \subsetneq \mathrm{SO}.$$

Idea of the proof

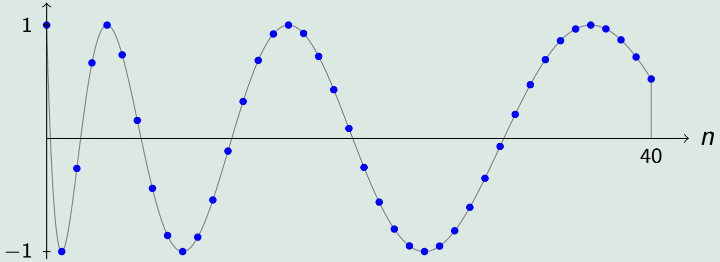
If $n \to \infty$ and $m = n + 1$, then $\frac{m}{n} \to 1$. Therefore $\mathrm{SO}_1 \subseteq \mathrm{SO}$.

To show that $\mathrm{SO}_1 \neq \mathrm{SO}$, consider the sequence

$$x_n = \cos(\pi\sqrt{n}).$$

Example of a sequence $\in \mathrm{SO} \setminus \mathrm{SO}_1$

$$x_n = \cos\left(\pi\sqrt{n}\right)$$



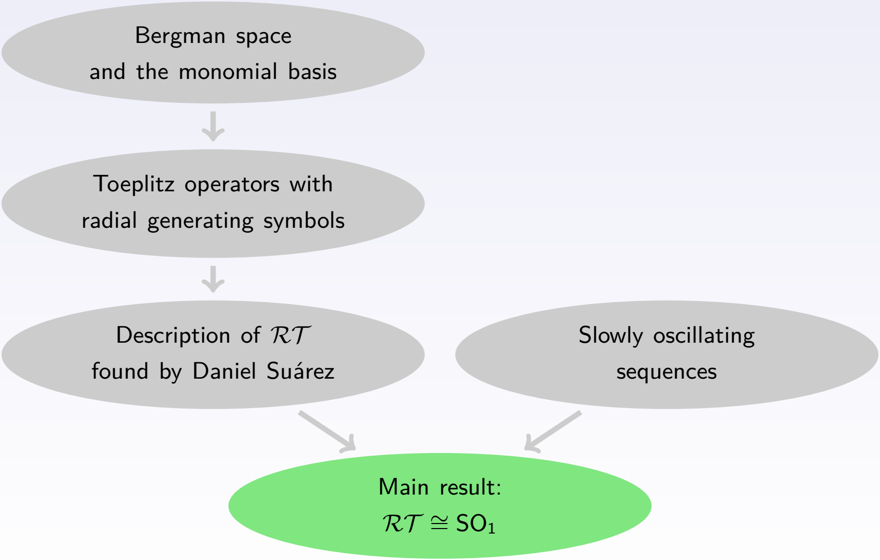
This sequence belongs to SO:

$$|x_{n+1} - x_n| \to 0 \qquad \text{as} \qquad n \to \infty,$$

but doesn't belong to SO_1:

$$|x_m - x_n| \not\to 0 \qquad \text{as} \qquad \frac{m}{n} \to 1.$$

Contents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Theorem

$$\mathcal{A} = \mathrm{SO}_1\,.$$

As a corollary,

$$\mathcal{RT} \cong \mathrm{SO}_1\,.$$

Idea of the proof

The result of Suárez says that d_1 is dense in $\mathcal{A}$.

We prove directly that d_1 is dense in SO_1. So,

$$\mathcal{RT} \cong \mathcal{A} = \overline{d_1} = \mathrm{SO}_1\,.$$