UNIVERSITY OF PAVOL JOZEF ŠAFÁRIK IN KOŠICE

Vertical Toeplitz operators acting on poly-analytic Bergman spaces over the upper half-plane

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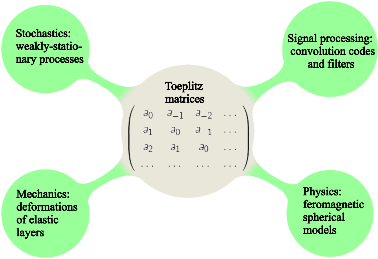
ESFM IPN, México City

Toeplitz operators



OTTO TOEPLITZ (1881–1940)

Basic idea: Let X be a Hilbert space, $Y \subset X$ and $P : X \to Y$ be an orthogonal projection. If A is a bounded operator on X, then Toeplitz operator with symbol A

$$T_A : x \in X \mapsto P(Ax) \in Y$$

is a compression of operator A to subspace Y, and it is an important operator theory model.

$T_A : x \in X \mapsto P(Ax) \in Y$

- Hardy space $Y \subset L_2(\mathbb{T})$ ... well-understood case (20. century – L. COBURN, P. HALMOS, I. GOHBERG, N. KRUPNIK, ...)
- Bergman space of $L_2(D)$-analytic functions on $D \subset \mathbb{C}$ ... S. AXLER, Ž. ČUČKOVIĊ, S. GRUDSKY, B. KORENBLUM, K. STROETHOFF, N. VASILEVSKI, ...
- Fock (Segal-Bargmann) space $Y \subset L_2(\mathbb{C}^d, \pi^{-d}\, \mathrm{e}^{-z\cdot\overline{z}}\, \mathrm{d}v(z))$ ... W. BAUER, L. COBURN, R. DOUGLAS, M. ENGLIŠ, T. LE, K. ZHU, ...
- if $Y \subset L_2(\mathbb{R}^{2d}, \mathrm{d}x\mathrm{d}y)$ is a space of STFT of $L_2(\mathbb{R}^d)$-functions, then T_a^{φ} is a Gabor-Toeplitz operator ... E. CORDERO, I. DAUBECHIES, H. FEICHTINGER, K. GRÖCHENIG, F. DE MARI, J. TOFT, ...
- if $Y \subset L_2(\mathbb{G}, u^{-2}\mathrm{d}u\mathrm{d}v)$ is the Calderón space of $L_2(\mathbb{R})$-functions, then T_a^{ψ} is a Calderón-Toeplitz operator ... K. NOWAK, L. PENG, R. ROCHBERG

Realization in time-frequency context

$$T_a^{\Psi} f = \int_{\mathbb{G}} a(\xi)(W_\Psi f)(\xi)\Psi_\xi \,\mathrm{d}\xi, \quad f \in L_2(\mathbb{R})$$

✓ Short-time Fourier transform (STFT)

$$(V_\varphi f)(q,p) = \int_{\mathbb{R}} f(x)\,\overline{\varphi(t-q)}\,\mathrm{e}^{-2\pi i p x}\,\mathrm{d}x, \quad p,q \in \mathbb{R}$$

✓ Continuous wavelet transform (CWT)

$$(W_\psi f)(u,v) = \frac{1}{\sqrt{u}} \int_{\mathbb{R}} f(x)\,\overline{\psi\left(\frac{x-v}{u}\right)}\,\mathrm{d}x, \quad u > 0, v \in \mathbb{R}$$

✓ Continuous Stockwell transform (CStT)

$$(S_\varphi f)(b,\xi) = \frac{|\xi|}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\,\overline{\varphi(\xi(x-b))}\,\mathrm{e}^{-ix\xi}\,\mathrm{d}x, \quad b \in \mathbb{R}, \xi \in \mathbb{R}\setminus\{0\}$$

✓ Localization operators related to shearlet transforms, ridgelet transforms, windowed Hankel transforms, etc.

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Bergman space over the upper half-plane Π

- the Bergman space ... $\mathcal{A}^2(\Pi) := \{f \in L_2(\Pi); \partial_{\overline{z}} f = 0\}$
- the evaluation functional at the point $z \in \Pi$

$$\mathrm{ev}_z : \mathcal{A}^2(\Pi) \to \mathbb{C}, \qquad \mathrm{ev}_z(f) := f(z)$$

  is bounded, thus by Riesz–Fréchet theorem there exists a function $K_{\Omega,z} \in \mathcal{A}^2(\Pi)$ such that

$$\forall (f \in \mathcal{A}^2(\Pi)) \quad \langle f, K_{\Pi,z} \rangle = f(z);$$

- the function $K_{\Pi,z}$ is called the Bergman kernel corresponding to Π and z;
- the formula $\langle f, K_{\Pi,z} \rangle = f(z)$ is called the reproducing property of the Bergman kernel;
- the orthogonal Bergman projection $P_\Pi : L_2(\Pi) \to \mathcal{A}^2(\Pi)$ takes the form

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Isomorphism $R\colon \mathcal{A}^2(\Pi) \to L_2(\mathbb{R}_+)$

Vasilevski (1999) constructed an isometric isomorphism $R\colon \mathcal{A}^2(\Pi) \to L_2(\mathbb{R}_+)$ given by

$$(R\varphi)(x) := \frac{\sqrt{x}}{\sqrt{\pi}} \int_\Pi \varphi(w)\, e^{-i\overline{w}x}\, d\mu(w).$$

The operator R is unitary, and its inverse $R^*\colon L_2(\mathbb{R}_+) \to \mathcal{A}^2(\Pi)$ is given by

$$(R^*f)(z) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}_+} \sqrt{\xi}\, f(\xi)\, e^{iz\xi}\, d\xi.$$

The operators R and R^* serve us to diagonalize the vertical operators.

N. L. VASILEVSKI: On the structure of Bergman and poly-Bergman spaces. *Integr. Equ. Oper. Theory* **33**(4) (1999), 471–488

Vertical operators on $\mathcal{A}^2(\Pi)$

- for $h \in \mathbb{R}$ denote by $\tau_h : L_2(\Pi) \to L_2(\Pi)$ the horizontal translation operator $(\tau_h f)(w) := f(w - h)$
- A bounded linear operator $S\colon \mathcal{A}^2(\Pi) \to \mathcal{A}^2(\Pi)$ is said to be vertical if $(\forall h \in \mathbb{R})\ \ \tau_h S = S\tau_h$.

Criterion for vertical operators

Let $S\colon \mathcal{A}^2(\Pi) \to \mathcal{A}^2(\Pi)$ be a bounded linear operator. Then the following conditions are equivalent:

(a) S is vertical;

(b) S is diagonalized by the unitary operator R, i.e.,

$$(\exists \sigma \in L_\infty(\mathbb{R}_+))\ \ RSR^* = M_\sigma.$$

(c) The Berezin transform of S depends on the imaginary part only, i.e., $\mathcal{B}(S)(u + \mathrm{i}\,v) = \mathcal{B}(S)(\mathrm{i}\,v)$ for all $u \in \mathbb{R}$ and $v > 0$.

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- For $g \in L_\infty(\Pi)$ we denote by M_g the multiplication by g in $L_2(\Pi)$:

$$M_g : L_2(\Pi) \to L_2(\Pi), \quad M_g(f) := fg.$$

- The Toeplitz operator with generating symbol g is defined as the compression of M_g to the subspace $\mathcal{A}^2(\Pi)$:

$$T_g f := P_\Pi M_g(f).$$

- Due to the reproducing formula T_g can be written in the following integral form using the Bergman kernel:

$$(T_g f)(z) = \langle M_g(f), K_{\Pi,z} \rangle = \int_\Pi f(w) g(w) \overline{K_{\Pi,z}(w)} \, \mathrm{d}\mu(w).$$

- General properties of Toeplitz operators on the Bergman space have been studied by Ahern, Axler, Coburn, Čučković, Rao, Zheng, Zhu, etc.

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Criterion for vertical Toeplitz operators

Let $g \in L_\infty(\Pi)$. Then T_g is vertical if and only if g is vertical.

Diagonalization of vertical Toeplitz operators

Let $a \in L_\infty(\mathbb{R}_+)$ and denote by $\tilde{a}$ the vertical extension of a, i.e.,

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Then $RT_{\tilde{a}}R^* = M_{\gamma_a}$, where

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$\mathcal{T} :=$ C*-algebra$(\mathfrak{T})$	$\mathcal{G} :=$ C*-algebra$(\mathfrak{G})$

Fact: The C*-algebras $\mathcal{T}$ and $\mathcal{G}$ are isometrically isomorphic. Moreover, $\mathcal{T}$ is commutative.

Problem: Find an independent description of $\mathcal{G}$!

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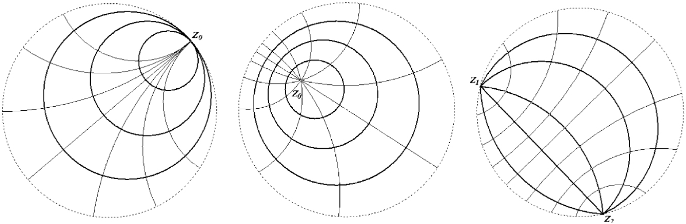
Intermezzo: C^*-algebras of TOs

Vasilevski characterization theorem

A Toeplitz operator algebra is commutative on each weighted Bergman space if and only if the corresponding symbol set consists of (smooth) functions which are constant on the cycles of a pencil of hyperbolic geodesics.

✓ **elliptic** – realized by radial symbols (depending on $|z|$ on $\mathbb{D}$),
✓ **parabolic** – realized by vertical symbols (depending on $\mathrm{Im}(z)$ on Π),
✓ **hyperbolic** – realized by angular symbols (depending on $\arg z$ on Π)



Intermezzo: C*-algebras of TOs

An isometric characterization of the elliptic, parabolic and hyperbolic case commutative C*-algebra of TOs:

(i) the C*-algebra generated by the set of TOs with bounded radial symbols is isometrically isomorphic to the C*-algebra VSO($\mathbb{N}$);

GRUDSKY, S. M., MAXIMENKO, E. A., VASILEVSKI, N. L.: Radial Toeplitz operators on the unit ball and slowly oscillating sequences. *Commun. Math. Anal.* **14**(2) (2013), 77–94.

(ii) the C*-algebra generated by the set of TOs with bounded vertical symbols is isometrically isomorphic to the C*-algebra VSO($\mathbb{R}_+$);

HERRERA YAÑEZ, C., MAXIMENKO, E. A., VASILEVSKI, N. L.: Vertical Toeplitz operators on the upper half-plane and very slowly oscillating functions. *Integr. Equ. Oper. Theory* **77**(2) (2013), 149–166.

(iii) the C*-algebra generated by the set of TOs with bounded angular symbols is isometrically isomorphic to the C*-algebra VSO($\mathbb{R}$).

ESMERAL, K., MAXIMENKO, E. A., VASILEVSKI, N. L.: C*-algebra generated by angular Toeplitz operators on the weighted Bergman spaces over the upper half-plane. *Integr. Equ. Oper. Theory* **83**(3) (2015), 413–428.

Poly-analytic Bergman spaces over Π

- the set $\mathcal{A}_n^2(\Pi) := \left\{ f \in L_2(\Pi); \partial_{\overline{z}}^n f = 0 \right\}$ is called the *n*-analytic Bergman space over Π;
- the set $\mathcal{A}_{(n)}^2(\Pi) := \mathcal{A}_n^2(\Pi) \ominus \mathcal{A}_{n-1}^2(\Pi)$ is called the true-*n*-analytic Bergman space over Π;
- analogically we define the *n*-anti-analytic and true-*n*-anti-analytic Bergman spaces over Π as follows

$$\widetilde{\mathcal{A}}_n^2(\Pi) := \{ f \in L_2(\Pi); \partial_z^n f = 0 \}$$

$$\widetilde{\mathcal{A}}_{(n)}^2(\Pi) := \mathcal{A}_n^2(\Pi) \ominus \mathcal{A}_{n-1}^2(\Pi)$$

- Vasilevski decomposition result

$$L_2(\Pi) = \bigoplus_{n=1}^{\infty} \mathcal{A}_{(n)}^2(\Pi) \oplus \bigoplus_{n=1}^{\infty} \widetilde{\mathcal{A}}_{(n)}^2(\Pi)$$

N. L. VASILEVSKI: On the structure of Bergman and poly-Bergman spaces. *Integr. Equ. Oper. Theory* **33**(4) (1999), 471–488

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(Un)expected time-scale approach

For $n \in \mathbb{Z}_+$ consider (a family of) wavelets

$$\hat{\psi}^{(n)}(\xi) := \chi_+(\xi)\sqrt{2\xi}\, \mathrm{e}^{-\xi} L_n(2\xi), \quad \hat{\bar{\psi}}^{(n)}(\xi) := \hat{\psi}^{(n)}(-\xi),$$

where

$$L_n(y) = \frac{\mathrm{e}^y}{n!} \frac{\mathrm{d}^n}{\mathrm{d}y^n} \left(\mathrm{e}^{-y} y^n\right) = \sum_{i=0}^{n} \binom{n}{i} \frac{(-y)^i}{i!}, \quad y \in \mathbb{R}_+.$$

Then the spaces of wavelet transforms

$$A^{(n)} = \{[W_n f](\zeta) = (f * \psi_v^{(n)})(u);\ f \in H_2(\mathbb{R})\}$$
$$\bar{A}^{(n)} = \{[W_{\bar{n}} f](\zeta) = (f * \psi_v^{(n)})(u);\ f \in H_2(\mathbb{R})^\perp\}$$

are exactly the true-$(n+1)$-analytic, resp. true-$(n+1)$-anti-analytic Bergman spaces over Π!

L. D. ABREU: Super-wavelets versus poly-Bergman spaces. *Integr. Equ. Oper. Theory* **73**(2) (2012), 177–193

L. D. ABREU, H. G. FEICHTINGER: *Function spaces of polyanalytic functions*. In: "Harmonic and Complex Analysis and its Applications", A. Vasil'ev (Ed.), Springer, 2014, pp. 1–38.

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Vertical TOs in poly-analytic setting

Diagonalization of vertical Toeplitz operators on $\mathcal{A}^2_{(n)}$

Let $a \in L_\infty(\mathbb{R}_+)$. Then the vertical Toeplitz operator $T^{(n)}_{\widetilde{a}}$ acting on $\mathcal{A}^2_{(n)}$ is unitarily equivalent to the operator $M_{\gamma_{a,n}}$ with

$$\gamma_{a,n}(x) = \int_{\mathbb{R}_+} a\left(\frac{t}{2x}\right) \mathrm{e}^{-t} L_n^2(t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+.$$

- Task: Describe $\mathfrak{G}_n = \{\gamma_{a,n};\, a \in L_\infty(\mathbb{R}_+)\}$!
- the C*-algebra generated by vertical TOs acting on $\mathcal{A}^2(\Pi)$ is commutative and isometrically isomorphic to the C*-algebra generated by $\mathfrak{G}_0$

VASILEVSKI, N. L.: *Commutative Algebras of Toeplitz Operators on the Bergman Space*, Birkhäuser Basel, 2008.

- $\mathfrak{G}_0$ is dense in the C*-algebra VSO$(\mathbb{R}_+)$

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Thema II: Approximate invertibility

Algebras: Let $\mathcal{A}$ be a normed complex vector space and at the same time an algebra. It is said that $\mathcal{A}$ is a normed algebra if the norm in $\mathcal{A}$ is *submultiplicative*:

$$(\forall a, b \in \mathcal{A}) \ \ \|ab\| \leq \|a\| \, \|b\|.$$

A normed algebra $\mathcal{A}$ is called a Banach algebra if $\mathcal{A}$ is *complete* w.r.t. the distance induced by the norm.

Definition (approximate identity)

Let $\mathcal{A}$ be a Banach algebra. A sequence $(e_j)_{j\in\mathbb{N}}$ in $\mathcal{A}$ is called an approximate identity in $\mathcal{A}$ if for every $a \in \mathcal{A}$

$$\lim_{j\to\infty} ae_j = a, \qquad \lim_{j\to\infty} e_j a = a.$$

Definition (right approximately invertible elements)

Let $x \in \mathcal{A}$. We say that x is right approximately invertible if there exists a sequence $(u_j)_{j\in\mathbb{N}}$ such that $(xu_j)_{j\in\mathbb{N}}$ is an approximate identity in $\mathcal{A}$.

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Approximate invertibility – a similar concept

THATTE, A.D., BHATT, S.J.: On topolizing invertibility. *Indian J. Pure Appl. Math.* **15**(12) (1984), 1308–1312.

Definition (right topologically invertible elements)

An element x in a unital algebra $\mathcal{A}$ (with unit e) is called right topologically invertible if there exists a sequence $(r_j)_{j\in\mathbb{N}} \in \mathcal{A}$ such that the sequence$(xu_j)_{j\in\mathbb{N}}$ converges to e.

Example: Aren's algebra

$$L^W([0,1]) := \bigcap_{1\le p\le+\infty} L_p([0,1])$$

is a unital complete metrizable algebra with pointwise operations and the topology of L_p-convergence for each $1 \le p < +\infty$. Then $f(x) = x$ is not invertible in $L^W([0,1])$, but it is topologically invertible, since

$$\exists g_j(x) = \chi_{[1/j,1]}(x)\frac{1}{x} \in L^W([0,1]) \quad : \quad \|fg_j - 1\|_p \to 0$$

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Dirac sequences

Now we consider $L_1(\mathbb{R})$ with the convolution operation $*$ making it a non-unital commutative algebra. For every $f \in L_1(\mathbb{R})$ denote by $\widehat{f}$ the Fourier transform of f.

Definition

A sequence $(e_j)_{j\in\mathbb{N}}$ in $L_1(\mathbb{R})$ is a Dirac sequence if:

(i) $e_j(x) \geq 0$ for every $x \in \mathbb{R}$, $j \in \mathbb{R}$;

(ii) $\int_{\mathbb{R}} e_j(x)\,\mathrm{d}x = 1$ for every $j \in \mathbb{N}$;

(iii) for every $\delta > 0$,

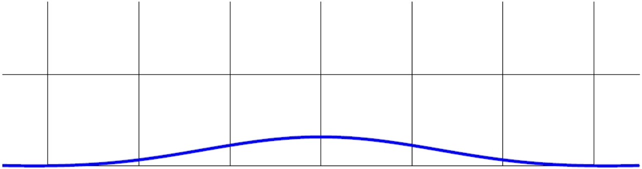
$$\lim_{j\to\infty} \int_{|x|\geq\delta} e_j(x)\,\mathrm{d}x = 0.$$

It is known that every Dirac sequence is an approximate identity in $L_1(\mathbb{R})$.

Dirac sequences – example

$$e_j(x) = \frac{(\sin(jx))^2}{\pi j x^2}.$$

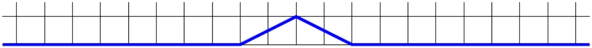
$j = 1$



In this example the support of $\widehat{e_j}$ is compact for every $j \in \mathbb{N}$:

$$\widehat{e_j}(t) = \begin{cases} 1 - \frac{|t|}{2j}, & |t| \leq 2j; \\ 0, & \text{otherwise.} \end{cases}$$

$j = 1$



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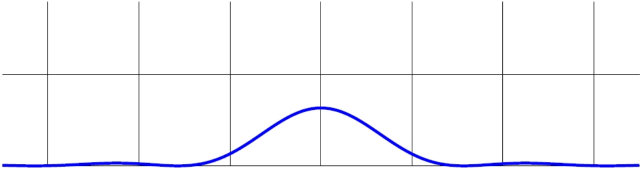
$j = 2$



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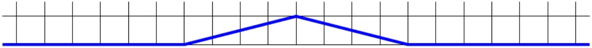

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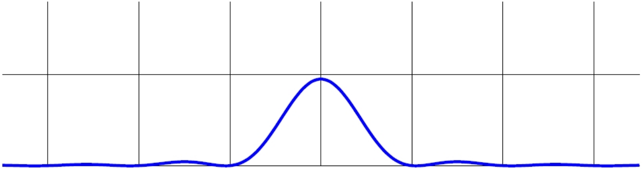
$$e_j(x) = \frac{(\sin(jx))^2}{\pi j x^2}.$$

$j = 3$



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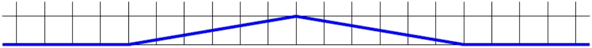

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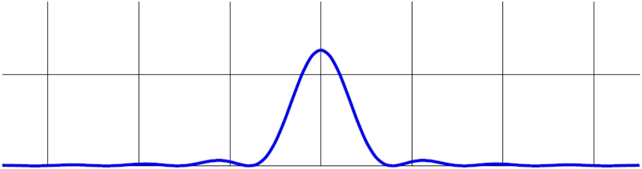

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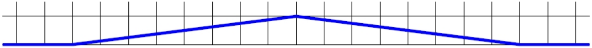

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$$e_j(x) = \frac{(\sin(jx))^2}{\pi j x^2}.$$

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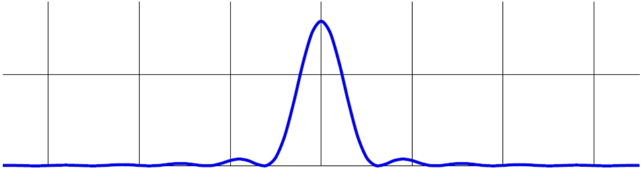

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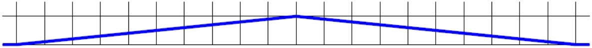

Approximate invertibility in $\mathcal{A} = L_1(\mathbb{R})$

Wiener's Division Lemma

Let $f, g \in L_1(\mathbb{R})$ such that

(i) $\mathrm{supp}(\widehat{f})$ is compact;

(ii) $\widehat{g}(x) \neq 0$ for every $x \in \mathrm{supp}(\widehat{f})$.

Then there exists $h \in L_1(\mathbb{R})$ such that $f = g * h$.

Criterion for approximate invertibility in $L_1(\mathbb{R})$

Then the following conditions are equivalent:

(a) f is approximately invertible in $L_1(\mathbb{R})$;

(b) $f * L_1(\mathbb{R})$ is dense in $L_1(\mathbb{R})$;

(c) $\widehat{f}(t) \neq 0$ for every $t \in \mathbb{R}$.

(c)⇒(a): Suppose that $\widehat{f}(t) \neq 0$ for every $t \in \mathbb{R}$. Let $(e_j)_{j\in\mathbb{N}}$ be a Dirac sequence with $\mathrm{supp}(\widehat{e_j}) \in \mathcal{K}$. For every $j \in \mathbb{R}$, by Wiener's Division Lemma, $\exists g_j \in L_1(\mathbb{R})$ such that $e_j = f * g_j$. Therefore, f is approximately invertible in $L_1(\mathbb{R})$.

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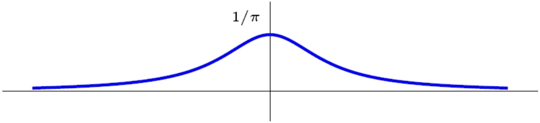
(c)⇒(a): Suppose that $\widehat{f}(t) \neq 0$ for every $t \in \mathbb{R}$. Let $(e_j)_{j\in\mathbb{N}}$ be a Dirac sequence with $\text{supp}(\widehat{e_j}) \in \mathcal{K}$. For every $j \in \mathbb{R}$, by Wiener's Division Lemma, $\exists g_j \in L_1(\mathbb{R})$ such that $e_j = f * g_j$. Therefore, f is approximately invertible in $L_1(\mathbb{R})$.

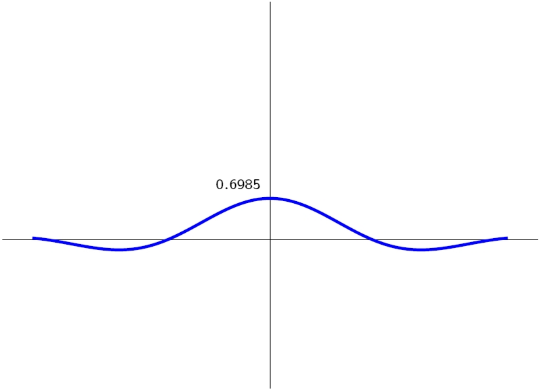
Approximate invertibility in $\mathcal{A} = L_1(\mathbb{R})$

$$f(x) = \frac{1}{\pi(1+x^2)}$$



h_1

$f * h_1 = e_1$



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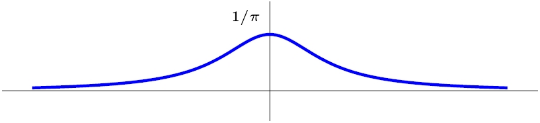
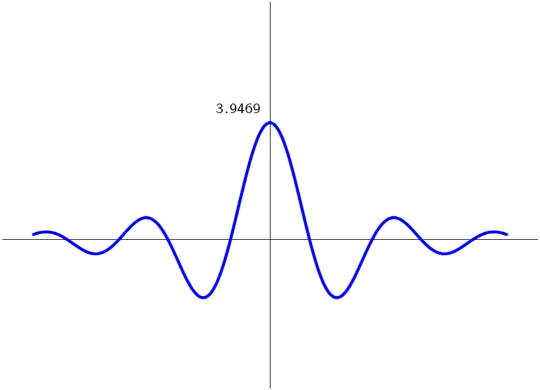


h_2

$f * h_2 = e_2$



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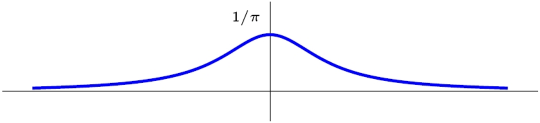
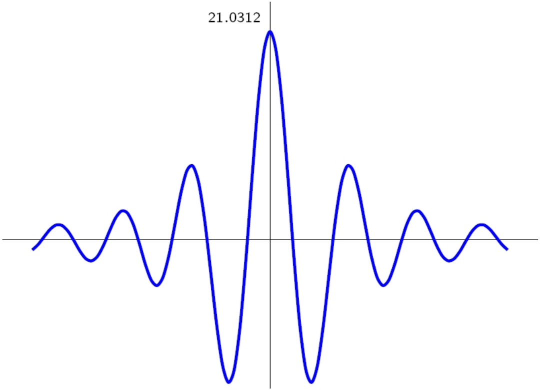


h_3

$f * h_3 = e_3$



Return back to the problem

- convolution algebra $L_1(\mathbb{R})$ does not have a unit;
- $(u_j)_{j\in J}$ is an approximative identity in a commutative Banach algebra $\mathcal{A}$, if $(au_j)_{j\in J} \to a$ for each $a \in \mathcal{A}$
- $a \in \mathcal{A}$ is approximatively invertible, if there exists $(b_j)_{j\in J}$ such that $(ab_j)_{j\in J}$ is an approximative identity in $\mathcal{A}$;

Approximative deconvolution on the real line

Let $K \in L_1(\mathbb{R})$ such that $\widehat{K}$ does not vanish on $\mathbb{R}$. Then there exists a sequence $(\varphi_j)_{j\in\mathbb{N}}$ in $L_1(\mathbb{R})$ such that $(K * \varphi_j)_{j\in\mathbb{N}}$ is a Dirac sequence.

Density of the image of the convolution operator

Let $K \in L_1(\mathbb{R})$. If $\widehat{K}$ does not vanish on $\mathbb{R}$, then the set

$$\mathfrak{G} = \{K * a;\ a \in L_\infty(\mathbb{R})\}$$

is a dense subset of $C_u(\mathbb{R})$.

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Finale: Vertical TOs in poly-analytic setting

Density of the set of spectral functions

For each $n \in \mathbb{Z}_+$ the set $\mathfrak{G}_n$ is dense in $\mathrm{VSO}(\mathbb{R}_+)$.

Very slowly oscillating functions:

A bounded function $f : \mathbb{R}_+ \to \mathbb{C}$ is very slowly oscillating iff

I. $\lim_{\frac{x}{y} \to 1} \left| f(x) - f(y) \right| = 0$;

II. the composition $f \circ \exp : \mathbb{R} \to \mathbb{C}$ is uniformly continuous with respect to the usual metric $d(x, y) = |x - y|$ on $\mathbb{R}$;

III. f is uniformly continuous with respect to the logarithmic metric $\rho(x, y) = |\ln x - \ln y|$ on $\mathbb{R}_+$;

Or, equivalently

$$\mathrm{VSO}(\mathbb{R}_+) := \left\{ f \in L_\infty(\mathbb{R}_+) : \lim_{\delta \to 0^+} \omega_{\rho,\sigma}(\delta) = 0 \right\},$$

where $\omega_{\rho,f} : [0, +\infty) \to [0, +\infty]$ is the modulus of continuity of f w.r.t. ρ, i.e.,

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Substitution $u = \mathrm{e}^x/2$ transforms the function

$$\gamma_{a,n}(u) = \int_{\mathbb{R}_+} a\left(\frac{t}{2u}\right)\ell_n^2(t)\,\mathrm{d}t, \quad u \in \mathbb{R}_+$$

to the convolution

$$\Gamma_{b,n}(x) = \gamma_{a,n}(\mathrm{e}^x/2) = \int_{\mathbb{R}} b(x-y)K_n(y)\,\mathrm{d}y$$

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II. step in the proof of density in poly-analytic case: Find the "right" expression for the Fourier transform of K_n, i.e., starting from the Howell formula

$$L_n^2(x) = \frac{1}{2^{2n}} \sum_{r=0}^{n} \alpha_r(n) L_{2r}(2x), \quad \text{where} \quad \alpha_r(n) := \binom{2n-2r}{n-r}\binom{2r}{r}$$

we come to the expressions

$$K_n(x) = \frac{1}{2^{2n}} \frac{\mathrm{e}^x}{\mathrm{e}^{\mathrm{e}^x}} \sum_{r=0}^{n} \alpha_r(n)\, L_{2r}(2\mathrm{e}^x)$$

$$\widehat{K}_n(t) = \frac{(-1)^n}{(n!)^2}\, \Gamma(1 - 2\pi\,\mathrm{i}\,t) P_n(2\pi t)$$

$$P_n(t) = \frac{(-1)^n\,(n!)^2}{2^{2n}} \sum_{r=0}^{n} \alpha_r(n)\; {}_2F_1(-2r, \mathrm{i}\,t; 1; 2)$$

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For each $n \in \mathbb{Z}_+$ the set $\mathfrak{G}_n$ is dense in $\mathrm{VSO}(\mathbb{R}_+)$.

III. step in the proof of density in poly-analytic case:
Understand the structure of zeros of complex polynomials P_n using the fact that

$$ {}_2F_1(-2r, \mathrm{i}\,t; 1; 2) = (-1)^r m_{2r}^{(1/2)}\left(t + \frac{\mathrm{i}}{2}, \frac{\pi}{2}\right). $$

Thus, each P_n is a linear combination of orthogonal polynomials

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$$ Q_n(x) = 4^n P_n\left(\frac{x - \mathrm{i}}{2}\right) = (-1)^n (n!)^2 \sum_{r=0}^{n} (-1)^r \alpha_r(n) m_{2r}^{(1/2)}\left(\frac{x}{2}, \frac{\pi}{2}\right) $$

$$ M_n(x) = (2n)!\, m_{2n}^{(1/2)}\left(\frac{\sqrt{x}}{2}, \frac{\pi}{2}\right) $$

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$$R_n(x) = \sum_{r=0}^{n} \beta_r(n) M_r(x) \quad \text{with} \quad \beta_r(n) := (-1)^{n-r} \binom{n}{r}^2 (2n-2r)!$$

The polynomials R_n satisfy the recurrence relation

$$xR_n(x) = R_{n+1}(x) + A_n R_n(x) + B_n R_{n-1}(x)$$

with $A_n = 8n(n+1)+3$, $B_n = 16n^4$ and $R_{-1}(x) = 0$, $R_0(x) = 1$.

- Favard's theorem = the polynomials R_n are orthogonal and all their zeros are real and simple;
- zeros of Q_n belong to $\mathbb{R} \cup \mathrm{i}\,\mathbb{R}$;
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Vertical TOs in poly-analytic setting

Vasilevski (1999) constructed the unitary operator U which is an isometric isomorphism of $L_2(\Pi)$ under which

- the true-n-analytic Bergman space $\mathcal{A}^2_{(n)}$ is mapped onto $L_2(\mathbb{R}_+) \otimes L_{n-1}$, where L_n is the one-dimensional space generated by Laguerre function ℓ_n;
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Diagonalization of vertical Toeplitz operators on $\mathcal{A}^2_n$

Let $a \in L_\infty(\mathbb{R}_+)$. Then the vertical Toeplitz operator $T^n_{\tilde{a}}$ acting on $\mathcal{A}^2_n$ is unitarily equivalent to the matrix multiplication operator $\gamma^{a,n}$ acting on $(L_2(\mathbb{R}_+))^n$ where the matrix-valued function $\gamma^{a,n} = (\gamma^{a,n}_{i,j})$ is given by

$$\gamma^{a,n}_{i,j}(x) = \int_{\mathbb{R}_+} a\left(\frac{t}{2x}\right) \ell_i(t)\ell_j(t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+.$$

RAMÍREZ ORTEGA, J., SÁNCHEZ-NUNGARAY, A.: Toeplitz operators with vertical symbols acting on the poly-Bergman spacesof the upper half-plane. *Complex Anal. Oper. Theory* **9**(8) (2015), 1801–1817.

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RAMÍREZ ORTEGA, J., SÁNCHEZ-NUNGARAY, A.: Toeplitz operators with vertical symbols acting on the poly-Bergman spacesof the upper half-plane. *Complex Anal. Oper. Theory* **9**(8) (2015), 1801–1817.

Postludium: A final joke :)

Problem: find the value of the sum

$$\sum_{k=0}^{2n}(-1)^k\sum_{r=0}^{k}\binom{k}{r}\binom{n}{k-r}\binom{n}{r}$$

in dependence on the value of n.

The basic relation

$$\gamma_{a,n}(x) = \sum_{k=0}^{2n} A_k(n)\frac{2x}{k!}\int_0^{+\infty} a(v)(2xv)^k \mathrm{e}^{-2vx}\,\mathrm{d}v = \sum_{k=0}^{2n} A_k(n)\,\gamma^V_{a,k}(x).$$

- for the generating symbol $a \equiv 1$ on $\mathbb{R}_+$ we easily get $\gamma_{a,n} \equiv 1$ for each $n \in \mathbb{Z}_+$ and $\gamma^V_{a,k}(x) = 1$ for each $n \in \mathbb{Z}_+$

Solution to the problem:

$$\sum_{k=0}^{2n} A_k(n) = \sum_{k=0}^{2n}(-1)^k\sum_{r=0}^{k}\binom{k}{r}\binom{n}{k-r}\binom{n}{r} \equiv 1 \quad \text{for each } n \in \mathbb{Z}_+.$$

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Solution to the problem:

$$\sum_{k=0}^{2n}A_k(n)=\sum_{k=0}^{2n}(-1)^k\sum_{r=0}^{k}\binom{k}{r}\binom{n}{k-r}\binom{n}{r}\equiv 1 \quad \text{for each } n\in\mathbb{Z}_+.$$

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Solution to the problem:

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Thanks for Your patience and attention!