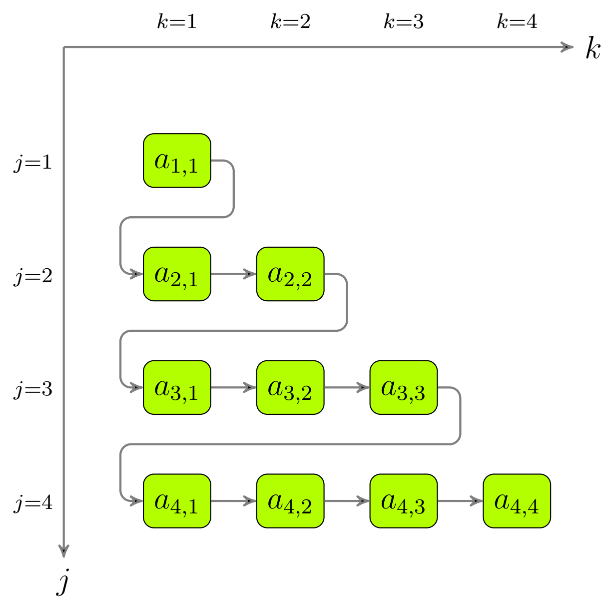
Lower triangular sums

Objectives. Learn to write lower triangular sums (starting with the index 1) in various manners: by rows, by columns and by diagonals.

Requirements. MATLAB, GNU Octave, Wolfram Mathematica or another programming language working with matrices and starting indices by 1.

Lower triangular summing by rows

1. Consider the sum corresponding to the picture. We are going to write in a short form the sum of all nodes in the order indicated by arrows.



First write all summands in the explicit form (joining summands by rows):

$$S = \Big(a_{1,1}\Big) + \Big(a_{2,1} + a_{2,2}\Big) + \Big(\qquad\qquad\qquad\qquad \Big) + \Big(\qquad\qquad\qquad\qquad\qquad\qquad \Big).$$

Rewrite each one of the little sums using the symbol $\sum$:

$$S = \sum_{k=1}^{1} a_{1,k} + \sum_{k=1}^{2} a_{2,k} + \sum \qquad + \sum \qquad .$$

Note that all summands have the same form. Pass to a double sum:

$$S = \sum_{j=} \sum_{k=} a_{j,k}.$$

2. Program (summing by rows the entries of a lower triangular matrix). Write a simple program that creates a random matrix and sums all its entries row by row. Here is an example of such a program in GNU Octave (substitute ??? by apropiate expressions):

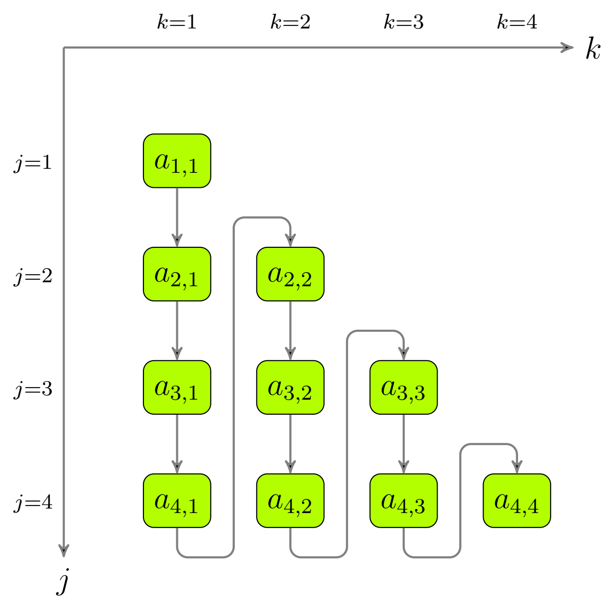
```
A = tril(1 + round(8 * rand(4, 4)));
disp(A);
s = 0;
for j = 1 : 4
    for k = ??? : ???
        s = s + A(j, k);
    endfor
endfor
disp("My answer:"); disp(s);
disp("Correct answer:"); disp(sum(sum(A)));
```

3. Lower triangular summing by rows. Generalize the formula to the case of a $n \times n$ matrix:

$$S = \sum_{j=} \sum_{k=} \qquad .$$

Lower triangular summing by columns

4. Write the sum corresponding to the picture. Now we are summing by columns.



First write all summands in the explicit form (joining them by columns):

$$S = \Big(a_{1,1} + a_{2,1} + a_{3,1} + a_{4,1}\Big) + \Big(\qquad\qquad\qquad\qquad \Big) + \Big(\qquad\qquad \Big) + \Big(\quad \Big).$$

Rewrite each one of the little sums using the symbol $\sum$:

$$S = \sum_{j=1}^{4} a_{j,1} + \sum \qquad + \sum \qquad + \sum \qquad .$$

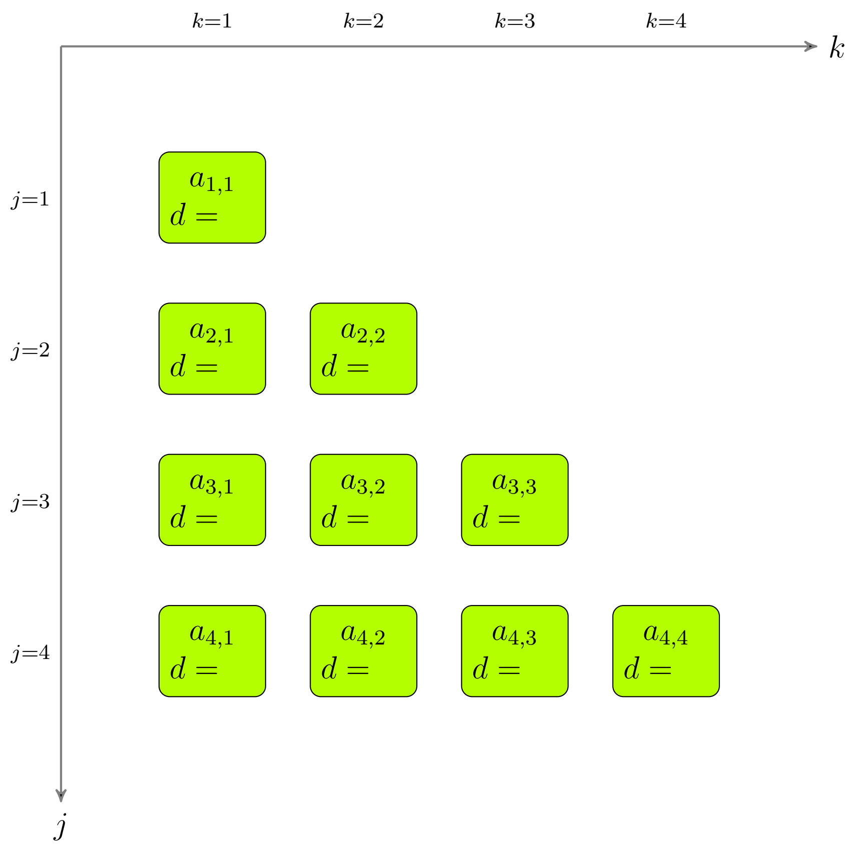
Note that all summands have the same form. Pass to a double sum:

$$S = \sum_{k=} \sum_{j=} a_{j,k}.$$

5. Write a program that creates a random lower triangular matrix and sums its entries by columns.

Diagonals of a lower triangular matrix

6. Calculate the difference $d := j - k$ for each entry $a_{j,k}$ of the following lower triangular matrix:



The entries satisfying $j = k$ form the *main diagonal* (or the *zeroth diagonal*) of the matrix; the entries satisfying $j - k = 1$ are on the *first diagonal*, and so on.

7. If $a_{j,k}$ belongs to the second diagonal, then j can be expressed through k in the following manner:

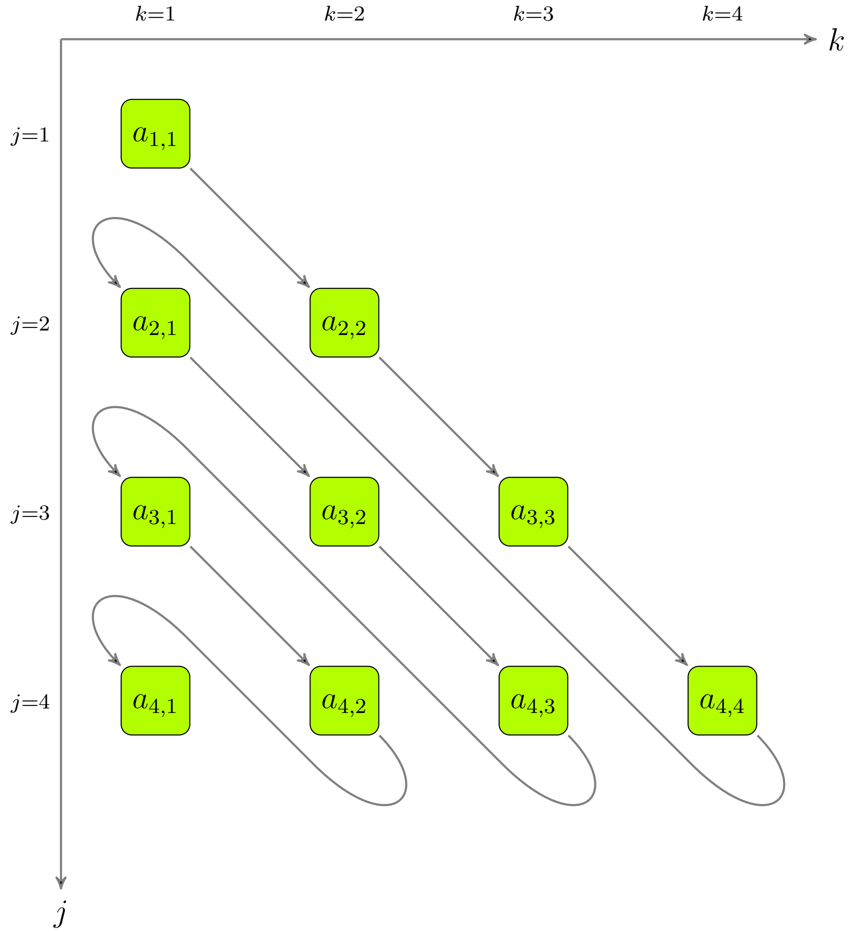
$$j = \underbrace{\qquad\qquad}_{?}.$$

8. Fix some $d \in \{0, 1, 2, 3\}$. If $a_{j,k}$ is on the diagonal with index d, then j can be expressed through k and d in the following manner:

$$j = \underbrace{\qquad\qquad}_{?}.$$

Lower triangular summing by diagonals

9. Write the sum corresponding to the picture. Now we are summing by diagonals.



First write all summands in the explicit form (joining them by diagonals):

$$S = \Bigl(a_{1,1} + a_{2,2} + a_{3,3} + a_{4,4}\Bigr) + \Bigl(\qquad\qquad\qquad \Bigr) + \Bigl(\qquad\qquad \Bigr) + \Bigl(\qquad \Bigr).$$

In each group the first index ("j") can be expressed through the second ("k") by a simple rule, so we can avoid j:

$$S = \sum_{k=1}^{4} a_{k,k} + \sum_{k=} \underbrace{\qquad}_{a_{?,k}} + \sum_{k=} \qquad + \sum_{k=} \qquad .$$

The first sum corresponds to $d = 0$, the second to $d = 1$, and so on:

$$S = \sum_{d=} \sum_{k=} \qquad .$$

Double lower triangular sum and formal changes of variables

10. Lower triangular summing by rows (recall). Recall the general formula for the lower triangular sum ($n \times n$ case) in which the terms are collected by rows:

$$S = \sum_{j=} \sum_{k=} a_{j,k}. \tag{1}$$

11. Summing by columns via an interchange of variables. Write two double inequalities that must satisfy j and k:

$$\underbrace{}_{?} \le j \le \underbrace{}_{?}, \qquad \underbrace{}_{?} \le k \le \underbrace{}_{?}.$$

Separate these two double inequalities in four simple inequalities:

$$j \ge \underbrace{}_{?}, \qquad j \le \underbrace{}_{?}, \qquad k \ge \underbrace{}_{?}, \qquad k \le \underbrace{}_{?}.$$

What is the maximum value (independent on j) that can take k?
What is the minimum value (independent on j) that can take k?
In other words, what are global limits of k?
Find the minimum and maximum values of k (independent on j):

$$\underbrace{}_{?} \le k \le \underbrace{}_{?}.$$

For each k fixed, find restrictions on j:

$$\underbrace{}_{?} \le j \le \underbrace{}_{?}.$$

Rewrite the sum S:

$$S = \sum_{k=} \sum_{j=} a_{j,k}.$$

12. Summing by diagonals via a formal change of variables. Write 4 inequalities that satisfy j and k in the sum (1):

$$j \geq \underbrace{\qquad}_{?}, \qquad j \leq \underbrace{\qquad}_{?}, \qquad k \geq \underbrace{\qquad}_{?}, \qquad k \leq \underbrace{\qquad}_{?}.$$

Find restrictions that satisfy the difference d:

$$\underbrace{\qquad\quad}_{?} \leq d \leq \underbrace{\qquad\quad}_{?}$$

Express j through d and k:

$$j = \underbrace{\qquad\quad}_{?}.$$

Substitute this expression into the equalities with j, then solve them for k (treat d as a parameter):

Rewrite the sum S as a double sum over d and k:

$$S = \sum_{d=} \sum_{k=} \underbrace{\qquad\qquad}_{a_{?,k}}.$$