Very slowly oscillating functions on the positive half-line

This text is a rough draft.

Objectives. Study the C^*-algebra of the bounded functions $f\colon \mathbb{R}_+ \to \mathbb{C}$ that are uniformly continuous with respect to the *logarithmic distance*

$$\rho(x,y) \coloneqq \bigl|\ln(x) - \ln(y)\bigr|.$$

Requirements. Bounded uniformly continuous functions on a metric space, logarithmic distance ρ and its properties, uniform deviation $\Omega_{\rho,f}$ with respect to the logarithmic distance.

Bounded uniformly continuous functions on a metric space (review)

In these exercises we suppose that (M,ρ) is a metric space.

Exercise 1 (uniform norm = supremum-norm)**.** Let $f\colon M \to \mathbb{C}$. Recall the definition of the supremum-norm:

$$\|f\|_\infty \coloneqq$$

Exercise 2 (uniform deviation of a function defined on a metric space)**.** Let (M,ρ) be a metric space and $f\colon M \to \mathbb{C}$. Recall the definition:

$$\Omega_{\rho,f}(\delta) \coloneqq$$

Exercise 3 (bounded uniformly continuous functions on a metric space)**.** Recall the definition:

$$\mathrm{UC_b}(M,\rho) \coloneqq \Bigl\{ f \in \mathbb{C}^M \colon \qquad \Bigr\}.$$

Exercise 4 (bounds for arithmetic operations)**.** Let $f, g \in \mathrm{UC_b}(M,\rho)$ and $\lambda \in \mathbb{C}$. Recall identities or upper bounds for the supremum-norms and uniform deviations of the functions $f+g$, λf, fg and $\overline{f}$:

$$\|f+g\|_\infty \le \qquad\qquad \Omega_{\rho,f+g}(\delta) \le$$

$$\|\lambda f\|_\infty = \qquad\qquad \Omega_{\rho,\lambda f}(\delta) =$$

$$\|fg\|_\infty \qquad\qquad \Omega_{\rho,fg}(\delta)$$

$$\|\overline{f}\|_\infty \qquad\qquad \Omega_{\rho,\overline{f}}(\delta)$$

Exercise 5. Let $f, g\colon M \to \mathbb{C}$ and $\delta > 0$. Write an upper bound for $\Omega_{\rho,f}(\delta)$ in terms of $\Omega_{\rho,g}(\delta)$ and $\|f-g\|_\infty$. That inequality can be used to prove that $\mathrm{UC_b}(M,\rho)$ is a closed subset of the set of all bounded functions $M \to \mathbb{C}$.

$$\Omega_{\rho,f}(\delta) \le$$

Exercise 6. $\mathrm{UC_b}(M,\rho)$ has the following properties. Indicate with arrows logical relations between of them.

1) $\mathrm{UC_b}(M,\rho)$ is closed with respect to the linear operations.

2) $\|\cdot\|_\infty$ is a norm.

3) $\mathrm{UC_b}(M,\rho)$ is a normed vector space.

4) $\mathrm{UC_b}(M,\rho)$ with the distance induced by the norm $\|\cdot\|_\infty$ is a complete metric space.

5) $\mathrm{UC_b}(M,\rho)$ is a Banach space.

6) $\mathrm{UC_b}(M,\rho)$ is closed with respecto to the multiplication.

7) $\|\cdot\|_\infty$ is submultiplicative.

8) The constant function 1 belongs to $\mathrm{UC_b}(M,\rho)$, and $\|1\|_\infty = 1$.

9) $\mathrm{UC_b}(M,\rho)$ is a unital Banach algebra.

10) $\mathrm{UC_b}(M,\rho)$ is closed with respect to the conjugation $f \mapsto \overline{f}$.

11) $\|\overline{f}\,f\|_\infty = \|f\|_\infty^2$.

12) $\mathrm{UC_b}(M,\rho)$ is a C^*-algebra.

Logarithmic distance on the positive half-line (review)

Exercise 7 (logarithmic distance on the positive half-line)**.** Recall the definition:

$$\rho(x, y) \coloneqq$$

Exercise 8. Let $x, y, z > 0$. Simplify:

$$\rho(zx, zy) =$$

Exercise 9 (δ-neighborhood of 1)**.**

$$\rho(x, 1) < \delta \qquad \Longleftrightarrow$$

Exercise 10 (δ-neighborhood of x)**.**

$$\rho(x, y) < \delta \qquad \Longleftrightarrow \qquad \underbrace{\qquad\qquad}_{?} < y < \underbrace{\qquad\qquad}_{?}$$

Exercise 11.

$$\rho(x, 1) \le \delta \qquad \Longleftrightarrow$$

Exercise 12.

$$\rho(x, y) \le \delta \qquad \Longleftrightarrow \qquad \underbrace{\qquad\qquad}_{?} \le y \le \underbrace{\qquad\qquad}_{?}$$

Uniform deviation of functions with respect to the logarithmic distance (review)

Exercise 13. Using the result of the Exercise 12 write $\Omega_{\rho,f}(\delta)$ as a double supremum:

$$\Omega_{\rho,f}(\delta) = \sup_{\underbrace{\qquad}_{?}} \; \sup_{\underbrace{\qquad\qquad}_{?}} \; \underbrace{\qquad\qquad}_{?}.$$

Exercise 14. Write $\Omega_{\rho,f}(\delta)$ in terms of the quotient y/x:

$$\Omega_{\rho,f}(\delta) = \sup_{\substack{x,y>0 \\ \underbrace{\qquad}_{?} \le \frac{y}{x} \le \underbrace{\qquad}_{?}}} \; \underbrace{\qquad\qquad}_{?}.$$

Definition 1 (standard δ-deviation of a function f near a point x)**.** Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ and $\delta > 0$. The $\Omega_{d,f,x}(\delta)$ is defined by

$$\Omega_{d,f,x}(\delta) \coloneqq \sup\big\{|f(x) - f(y)|\colon\; d(x,y) \le \delta\big\}.$$

Exercise 15. Let $f\colon \mathbb{R}_+ \to \mathbb{C}$, $\delta > 0$ and $x > 0$. Compare $\Omega_{d,f,x}(\delta)$ with $\Omega_{\rho,f}(\delta)$.

Definition of very slowly oscillating function

Exercise 16. Prove that

$$\mathrm{UC_b}(\mathbb{R}_+,\rho) = \big\{g\circ\exp\colon\ g\in\mathrm{UC_b}(\mathbb{R},d)\big\}.$$

Definition 2 (very slowly oscillating functions)**.** Denote by $\mathrm{VSO}(\mathbb{R}_+)$ the following set:

$$\mathrm{VSO}(\mathbb{R}_+) \coloneqq \mathrm{UC_b}(\mathbb{R}_+,\rho) = \big\{g\circ\exp\colon\ g\in\mathrm{UC_b}(\mathbb{R},d)\big\}.$$

Exercise 17. Prove that $\mathrm{VSO}(\mathbb{R}_+)$ is a subset of $C_b(\mathbb{R}_+)$.

Examples

Exercise 18. Determine whether $f\in\mathrm{VSO}(\mathbb{R}_+)$, where

$$f\colon\mathbb{R}_+\to\mathbb{C},\qquad f(x)\coloneqq\cos\left(\ln(x)\right).$$

Exercise 19. Determine whether $f\in\mathrm{VSO}(\mathbb{R}_+)$, where

$$f\colon\mathbb{R}_+\to\mathbb{C},\qquad f(x)\coloneqq\cos\left(\sqrt{x}\right).$$

Comparison with the C^*-algebra of the continuous functions having finite limits at 0 and $+\infty$

Definition 3 (a distance on $[0,+\infty]$ inducing the standard topology)**.** The standard topology on $[0,+\infty]$ can be induced by the distance

$$d_{[0,+\infty]}(x,y) := \bigl|\zeta(x)-\zeta(y)\bigr|,$$

where

$$\zeta(x) := \begin{cases} \frac{x}{x+1}, & x \in [0,+\infty); \\ 1, & x = +\infty. \end{cases}$$

Exercise 20. Let $x, y \in \mathbb{R}_+$. Compare $d_{[0,+\infty]}(x,y)$ with $\rho(x,y)$.

Exercise 21. Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ and $\delta > 0$. Compare:

$$\Omega_{d_{[0,+\infty]},f}(\delta) \quad \underbrace{\qquad}_{?} \quad \Omega_{\rho,f}(\delta).$$

Definition 4. Given a function $f\colon \mathbb{R}_+ \to \mathbb{C}$, the following conditions are equivalent:

(a) $f \in C(\mathbb{R}_+)$ and f have finite limits at 0^+ and $+\infty$.

(b) f can be prolonged to a continuous function $[0,+\infty] \to \mathbb{C}$.

(c) f can be prolonged to a uniformly continuous function $[0,+\infty] \to \mathbb{C}$.

(d) f is uniformly continuous with respect to the distance $d_{[0,+\infty]}$.

The set of these functions will be denoted by $C([0,+\infty])$.

Exercise 22. Compare $\mathrm{VSO}(\mathbb{R}_+)$ with $C([0,+\infty])$:

$$\mathrm{VSO}(\mathbb{R}_+) \qquad \underbrace{\qquad}_{?} \qquad C([0,+\infty]).$$