Vertical Toeplitz operators on the Bergman space on the upper half-plane

This text is just a stub.

Objectives. Denote by $\mathcal{A}^2(\Pi)$ the Bergman space of the analytic square integrable functions on the upper half-plane Π of the complex plane. Given a function $g \in L^\infty(\Pi)$ denote by T_g the Toeplitz operator with generating symbol g acting in the space $\mathcal{A}^2(\Pi)$.

Prove that T_g is invariant under horizontal shifts if and only if g depends only on the imaginary part of the argument.

Requirements. Bergman space of the analytic square integrable functions on the upper half-plane, Berezin transform, shift operator.

Exercise 1. Find the following theorem for Toeplitz operators in $\mathcal{A}^2(\mathbb{D})$: If $a \in L^\infty(\mathbb{D})$ and $T_a = 0$, then $a = 0$ almost everywhere. This theorem can be found in the book of Vasilevski or in other papers.

Exercise 2. Let $\phi\colon \mathbb{D} \to \Pi$ be a biholomorphism and $E \subset \mathbb{D}$ be a set of zero Lebesgue measure: $\mu(E) = 0$. Prove that $\mu(\phi(E)) = 0$.

Exercise 3. Try to translate the theorem cited in the Exercise 1 to the case of Toeplitz operators in $\mathcal{A}^2(\Pi)$.

Definition 1 (horizontal shift). Let $h \in \mathbb{R}$. Define $H_h \in \mathcal{L}(\mathcal{A}^2(\Pi))$ by

$$(H_h f)(w) := f(w - h).$$

Exercise 4. Let $h \in \mathbb{R}$ and $a \in L^\infty(\Pi)$. Express the product

$$H_{-h} T_a H_h$$

as T_b for some $b \in L^\infty(\Pi)$.

Exercise 5. Let $a \in L^\infty(\Pi)$ such that

$$\forall h \in \mathbb{R} \qquad T_a H_h = H_h T_a.$$

Prove that $a(w + h) = a(w)$ for all $h \in \mathbb{R}$ and for almost all $w \in \Pi$.
Prove that $a(w) = a(\mathrm{i}\,\Im(w))$ for almost all $w \in \Pi$.