Vertical Toeplitz operators on the Bergman space on the upper half-plane and very slowly oscillating functions on the positive half-line

This theme is planned to be written in some weeks. The actual text is just a rough draft.

Objectives. Prove that the set of functions

$$\Gamma \coloneqq \bigl\{ \gamma_a \colon\ a \in L^\infty(\mathbb{R}_+) \bigr\},$$

where

$$\gamma_a(x) \coloneqq 2x \int_0^{+\infty} a(v)\ \mathrm{e}^{-2vx}\ dv,$$

is dense in the algebra $\mathrm{VSO}(\mathbb{R}_+)$ of the very slowly oscillating functions:

$$\mathrm{VSO}(\mathbb{R}_+) \coloneqq \Bigl\{ f \colon \mathbb{R}_+ \to \mathbb{C} \colon \quad \sup_{x \in \mathbb{R}_+} |f(x)| < +\infty \quad \wedge \quad \lim_{\frac{x}{y} \to 1} |f(x) - f(y)| = 0 \Bigr\}.$$

Requirements. Definition of $\mathrm{VSO}(\mathbb{R}_+)$ through the logarithmic distance ρ:

$$\rho(x, y) \coloneqq \bigl| \ln(x) - \ln(y) \bigr|,$$

basic properties of the functions ϕ_n and ψ_n defined by the following formulas:

$$\phi_n(v) \coloneqq \frac{1}{\bigl((n-1)!\bigr)^2} \frac{d^{n-1}}{dv^{n-1}} \left(\mathrm{e}^{-v}\ v^{2n-1} \right), \qquad \psi_n(t) \coloneqq \int_0^{+\infty} \mathrm{e}^{-vt}\ \phi_n(v)\, dv.$$

Logarithmic distance on the positive half-line (review)

Exercise 1 (logarithmic distance)**.** Recall the definition of $\rho(x,y)$:

$$\rho(x,y) \coloneqq$$

Exercise 2. Prove that the logarithmic distance is dilation-invariant:

$$\rho(tx,ty) =$$

Exercise 3. Compare the logarithmic distance with another dilation-invariant distance:

$$\rho(x,y) \underbrace{\qquad}_{\leq \text{ or } \geq} \frac{|x-y|}{\max(x,y)}.$$

Exercise 4 (δ-neighborhood of 1)**.** Find the δ-neighborhood of 1 with respect to the distance ρ:

$$\rho(x,1) < \delta \qquad \Longleftrightarrow$$

Very slowly oscillating functions on the positive half-line (review)

Exercise 5 (uniform deviation of a function with respect to the logarithmic distance)**.** Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ and $\delta > 0$. Write the definition and various descriptions of $\Omega_{\rho,f}(\delta)$.

Exercise 6 (very slowly oscillating functions)**.** Write the definition of $\mathrm{VSO}(\mathbb{R}_+)$.

Special approximate unit (review)

Exercise 7. Recall the definition of the function ϕ_n:

$$\phi_n(v) =$$

Exercise 8. Recall an explicit formula for the function ϕ_n (without derivatives):

$$\phi_n(v) = \qquad \sum$$

Exercise 9. Calculate:

$$\lim_{v \to 0^+} \phi_n(v) = \qquad \qquad \lim_{v \to +\infty} \phi_n(v) =$$

Exercise 10. Recall the explicit formula for the function ψ_n:

$$\psi_n(t) =$$

Exercise 11. Calculate the integral:

$$\int_0^{+\infty} \psi_n(t)\, dt =$$

Exercise 12. Let $\delta > 0$. Calculate the limit:

$$\lim_{n \to \infty} \int_0^{\mathrm{e}^{-\delta}} \psi_n(t)\, dt =$$

Exercise 13. Let $\delta > 0$. Make the change of variables $s = \dfrac{1}{t}$ in the following integral:

$$\int_{e^\delta}^{+\infty} \psi_n(t)\,dt =$$

Exercise 14. Let $\delta > 0$. Calculate the limit:

$$\lim_{n\to\infty} \int_{e^\delta}^{+\infty} \psi_n(t)\,dt =$$

Definition 1. Let $\delta > 0$. Denote by $R_{n,\delta}$ the following integral:

$$R_{n,\delta} := \int\limits_{\substack{x>0\\ \rho(x,1)>\delta}} \psi_n(t)\,dt.$$

Exercise 15. Write $R_{n,\delta}$ as a sum of some integrals with explicit limits of integration:

$$R_{n,\delta} = \int \ \psi_n(t)\,dt \ + \int \ \psi_n(t)\,dt.$$

Exercise 16. Let $\delta > 0$. Calculate the limit:

$$\lim_{n\to\infty} R_{n,\delta} =$$

Gammas belong to VSO($\mathbb{R}_+$)

The following two exercises show that the following set Γ is a subset of VSO($\mathbb{R}_+$).

$$\Gamma \coloneqq \left\{\gamma_a\colon\ a \in L^\infty(\mathbb{R}_+)\right\}$$

In fact, these results motivated the definition of the class VSO($\mathbb{R}_+$).

Exercise 17. Let $x \in \mathbb{R}_+$. Calculate the integrals:

$$2x \int_{\mathbb{R}_+} \mathrm{e}^{-2xv}\, dv = \qquad\qquad\qquad 2x^2 \int_{\mathbb{R}_+} v\, \mathrm{e}^{-2xv}\, dv =$$

Exercise 18. Let $a \in L^\infty(\Pi)$. Prove that γ_a is bounded.

$$|\gamma_a(x)| \le$$

Exercise 19. Let $x, y \in \mathbb{R}_+$, $x < y$. Apply the Mean Value Theorem to the function $t \mapsto \mathrm{e}^{-t}$:

$$\mathrm{e}^{-x} - \mathrm{e}^{-y} \le \left(\sup_{x<t<y} \left(\mathrm{e}^{-t}\right)'\right)(y-x) = \left(\sup_{x<t<y} \underbrace{\qquad}_{?}\right)(y-x) = \underbrace{\qquad}_{?}(y-x).$$

Exercise 20. Let $a \in L^\infty(\Pi)$. Prove that γ_a is Lipschitz-continuous with respect to the logarithmic distance ρ.

Gammas are dense in $\mathrm{VSO}(\mathbb{R}_+)$

Exercise 21. Let $n \in \{1, 2, \ldots\}$, $x > 0$, $y > 0$. Using an appropiate change of variables express the following integral in terms of ψ_n:

$$2x \int_0^{+\infty} \frac{\phi_n(2vy)}{y}\, \mathrm{e}^{-2xv}\, dv =$$

Then use the formula for ψ_n and write the result in an explicit form:

$$2x \int_0^{+\infty} \frac{\phi_n(2vy)}{y}\, \mathrm{e}^{-2xv}\, dv =$$

Exercise 22. Let $\sigma \in \mathrm{VSO}(\mathbb{R}_+)$ and $n \in \{1, 2, \ldots\}$. Define $a_n \in L^\infty(\mathbb{R}_+)$ by

$$a_n(v) \coloneqq \int_0^{+\infty} \sigma(y)\, \frac{\phi_n(2vy)}{y}\, dy. \tag{1}$$

Applying Fubini's theorem and the result of Exercise 21 calculate γ_{a_n}:

$$\gamma_{a_n}(x) =$$

Exercise 23. Let $\sigma \in \mathrm{VSO}(\mathbb{R}_+)$ and $n \in \{1, 2, \ldots\}$. Define a_n by (1). Find an upper bound for $\|\gamma_{a_n} - \sigma\|_\infty$ in terms of $\Omega_{\rho,\sigma}$, $\|\sigma\|_\infty$ and $R_{n,\delta}$.

$$|\gamma_{a_n}(x) - \sigma(x)| \le$$

Exercise 24. Let $\sigma \in \mathrm{VSO}(\mathbb{R}_+)$ and $\varepsilon > 0$. Choose $n \in \{1, 2, \ldots\}$ such that the function a_n defined by (1) fulfill $\|\gamma_{a_n} - \sigma\|_\infty < \varepsilon$.

Exercise 25 (main result)**.** Prove that the set $\bigl\{\gamma_a \colon\ a \in L^\infty(\mathbb{R}_+)\bigr\}$ is dense in $\mathrm{VSO}(\mathbb{R}_+)$.