Multiplication operator on the space of square-summable sequences

Objectives. Study some basic properties of the multiplication operator on the space ℓ^2.

Requirements. Diagonal matrices, product of diagonal matrices, the space ℓ^2 of the square-summable sequences of complex numbers, the space ℓ^∞ of the bounded sequences of complex numbers, Kronecker's delta, sums with Kronecker's delta, the supremum and the infimum of a set of real numbers, the supremum and the infimum of a sequence of real numbers, the closure of a set in a metric space, bounded linear operators, the norm of a bounded linear operator, the adjoint (the Hermitian conjugate) of a bounded linear operator, the invertibility of a bounded linear operator, the spectrum of a bounded linear operator.

Denote by $\mathbb{N}_0$ the set of the natural numbers: $\mathbb{N}_0 \coloneqq \{0, 1, 2, \ldots\}$. The notation $\mathbb{C}^{\mathbb{N}_0}$ will be used for the set of all the sequences of complex numbers, that is, for the set of all the functions $\mathbb{N}_0 \to \mathbb{C}$.

Definition 1 (the space of the square-summable sequences of complex numbers)**.**

$$\ell^2 \coloneqq \ell^2(\mathbb{N}_0) \coloneqq \Big\{ x \in \mathbb{C}^{\mathbb{N}_0} \colon \underbrace{\sum_j \quad x_j \qquad\qquad}_{?} \Big\}.$$

The space ℓ^2 is a Hilbert space with respect to the inner product

$$\langle x, y \rangle \coloneqq \sum_{j \in \mathbb{N}_0} \overline{x_j}\, y_j.$$

This inner product is linear with respect to the second argument. Many authors define inner product to be linear with respect to the first argument.

Definition 2 (the space of the bounded sequences of complex numbers)**.**

$$\ell^\infty \coloneqq \ell^\infty(\mathbb{N}_0) \coloneqq \Big\{ x \in \mathbb{C}^{\mathbb{N}_0} \colon \underbrace{\qquad\qquad\qquad\qquad}_{?} \Big\}.$$

Canonical base of ℓ^2 (review)

Definition 3 (canonical base of ℓ^2)**.** For every $n \in \mathbb{N}_0$, denote by e_n the sequence

$$e_n := \big(\delta_{n,j}\big)_{j\in\mathbb{N}_0}.$$

In the following exercises we shall see that $(e_n)_{n\in\mathbb{N}_0}$ is an orthonormal base of ℓ^2.

Exercise 1. Write the sequence e_2:

$$e_2 = \big(\,\delta_{2,0}\,,\ \delta_{2,1}\,,\ \delta_{2,2}\,,\ \delta_{2,3}\,,\ \delta_{2,4}\,,\ \ldots\,\big) = \big(\underbrace{\qquad}_{?},\ \underbrace{\qquad}_{?},\ \underbrace{\qquad}_{?},\underbrace{\qquad}_{?},\ \underbrace{\qquad}_{?},\ \ldots\big).$$

Exercise 2. Write the following sequences:

$$-3e_0 = \big(\quad,\quad,\quad,\quad,\quad,\ \ldots\big),$$

$$7e_1 = \big(\quad,\quad,\quad,\quad,\quad,\ \ldots\big),$$

$$4e_0 - 5e_3 = \big(\quad,\quad,\quad,\quad,\quad,\ \ldots\big).$$

Exercise 3. Write the following sequences in terms of the basic elements e_n:

$$\big(0,0,0,5,0,\ldots\big) = \underbrace{\qquad\qquad\qquad}_{?},\qquad \big(2,0,-4,0,0,\ldots\big) = \underbrace{\qquad\qquad\qquad}_{?}.$$

Exercise 4 (orthonormality of the canonical base)**.** Let $m, n \in \mathbb{N}_0$. Calculate the product:

$$\langle e_m, e_n\rangle =$$

Exercise 5 (norm of an element of the canonical base)**.** Let $n \in \mathbb{N}_0$. Calculate the norm of e_n:

$$\|e_n\|_2 = \underbrace{\qquad\qquad}_{?}.$$

Exercise 6 (norm of a multiple of an element of the canonical base)**.** Let $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. Calculate the norm of λe_n:

$$\|\lambda e_n\|_2 = \underbrace{\qquad\qquad}_{?}.$$

Exercise 7. Let $x \in \ell^2$ and $n \in \mathbb{N}_0$. Calculate:

$$\langle e_n, x \rangle =$$

Exercise 8. Suppose that $x \in \ell^2$ and $\langle e_n, x \rangle = 0$ for all $n \in \mathbb{N}_0$. Then

$$\underbrace{\qquad\qquad\qquad}_{?}$$

According to general criteria of orthonormal basis in a Hilbert space, it implies that the sequence $(e_n)_{n \in \mathbb{N}_0}$ is an orthonormal base of ℓ^2.

Exercise 9. Let $x \in \ell^2$. Write the decomposition of x in the base $(e_n)_{n \in \mathbb{N}_0}$:

$$x = \sum_{n=0}^{\infty} \underbrace{\qquad\qquad}_{?} e_n.$$

Exercise 10. Let $x \in \ell^2$ and $m \in \mathbb{N}_0$ such that

$$\forall n \in \mathbb{N}_0 \setminus \{m\} \qquad \langle e_n, x \rangle = 0.$$

What can you say about x? Write x in terms of some basic elements and some inner products.

Multiplication operator: definition and examples

Definition 4 (component-wise product of sequences)**.** Let $x, y \in \mathbb{C}^{\mathbb{N}_0}$. Denote by xy the *component-wise product* of the sequences x and y:

$$\forall n \in \mathbb{N}_0 \qquad (xy)_n \coloneqq x_n y_n.$$

In other words,

$$xy \coloneqq \big(x_n y_n\big)_{n\in\mathbb{N}_0}.$$

Definition 5 (multiplication operator)**.** Let $a \in \ell^\infty$. Define $M_a \colon \ell^2 \to \ell^2$ by

$$M_a x \coloneqq ax,$$

that is,

$$\forall x \in \ell^2 \qquad \forall j \in \mathbb{N}_0 \qquad (M_a x)_j \coloneqq a_j x_j.$$

Exercise 11. Let $a \colon \mathbb{N}_0 \to \mathbb{C}$ be defined by

$$a \coloneqq \big(1 + (-1)^n\big)_{n\in\mathbb{N}_0} = (\underbrace{\qquad}_{?}, \underbrace{\qquad}_{?}, \underbrace{\qquad}_{?}, \underbrace{\qquad}_{?}, \underbrace{\qquad}_{?}, \ldots).$$

Consider

$$x \coloneqq \left(\frac{1}{n+1}\right)_{n\in\mathbb{N}_0} = \Big(\qquad, \qquad, \qquad, \qquad, \qquad, \ldots\Big).$$

Calculate $M_a x$:

$$M_a x = \Big(\qquad, \qquad, \qquad, \qquad, \qquad, \ldots\Big).$$

Exercise 12 (identity operator as a multiplication operator)**.** Denote by I the identity operator on the space ℓ^2:

$$\forall x \in \ell^2 \qquad I(x) \coloneqq x.$$

Find a sequence $a \in \ell^\infty$ such that $M_a = I$.

$$a = \Big(\qquad, \qquad, \qquad, \qquad, \qquad, \ldots\Big)$$

Multiplication operator acting on the canonical base of ℓ^2

Exercise 13. Let $a \in \ell^\infty$ be a general bounded sequence:

$$a \coloneqq \big(a_0, a_1, a_2, a_3, a_4, \ldots\big).$$

Write first components of the basic element e_3:

$$e_3 = (\qquad, \qquad, \qquad, \qquad, \qquad, \ldots).$$

Calculate $M_a e_3$:

$$M_a e_3 = (\qquad, \qquad, \qquad, \qquad, \qquad, \ldots).$$

Write the answer in terms of some basic elements: $M_a e_3 = \underbrace{\qquad\qquad}_{?}.$

Exercise 14. Let $a \in \ell^\infty$ and $n \in \mathbb{N}_0$. Write a general formula for $M_a e_n$ and prove it.

Exercise 15. Let $a \in \ell^\infty$ and $n \in \mathbb{N}_0$. Calcule the ℓ^2-norm of the sequence $M_a e_n$:

$$\|M_a e_n\|_2 = \underbrace{\qquad\qquad}_{?}.$$

Supremum and infimum (review)

Definition 6. Denote by $\overline{\mathbb{R}}$ the *extended real number line* $\mathbb{R} \cup \{-\infty, +\infty\}$ with the canonical order. The additional elements $-\infty$ and $+\infty$ satisfy

$$\forall \alpha \in \mathbb{R} \qquad -\infty < a; \qquad\qquad \forall \alpha \in \mathbb{R} \qquad \alpha < +\infty; \qquad\qquad -\infty < +\infty.$$

Exercise 16 (upper bound of a set of real numbers)**.** Let $A \subset \overline{\mathbb{R}}$ and $\beta \in \overline{\mathbb{R}}$. Recall the definition:

$$\beta \quad \text{is an upper bound of} \quad A \qquad \Longleftrightarrow \qquad \underbrace{\qquad\qquad\qquad\qquad}_{?}$$

Exercise 17. Let $A \subset \overline{\mathbb{R}}$ and $\gamma \in \overline{\mathbb{R}}$. Then,

$$\gamma \quad \text{is not an upper bound of} \quad A \qquad \Longleftrightarrow \qquad \underbrace{\qquad\qquad\qquad\qquad}_{?}$$

Exercise 18. Let $A \subset \overline{\mathbb{R}}$ and $\beta \in \overline{\mathbb{R}}$. What does mean the phrase "β is the supremum of A"? Write the definition using the concept of the upper bounds. (It is known that for every $A \subset \overline{\mathbb{R}}$ there exists a unique supremum in $\overline{\mathbb{R}}$. We shall not prove this fact here.)

$$\beta = \sup(A) \qquad \Longleftrightarrow$$

Exercise 19. Let $A \subset \overline{\mathbb{R}}$ and $\beta \in \overline{\mathbb{R}}$. What does mean the phrase "β is the supremum of A"? Write the answer in terms of the quantifications $\forall$ and $\exists$ and some of the inequalities $<, >, \leq, \geq$, without mentioning explicitly the concept of upper bounds.

$$\beta = \sup(A) \qquad \Longleftrightarrow \qquad \begin{cases} 1) \quad \forall \alpha \in A \quad \dots \\ \\ 2) \quad \forall \gamma < \beta \quad \dots \end{cases}$$

Exercise 20. Let $x = (x_n)_{n \in \mathbb{N}_0}$ be a sequence in $\mathbb{R}$ and $\beta \in \overline{\mathbb{R}}$. What does mean the phrase "β is the supremum of the sequence x"? Write the answer in terms of the quantifications $\forall$ and $\exists$ and some of the inequalities $<, >, \leq, \geq$.

$$\beta = \sup_{n \in \mathbb{N}_0} x_n \qquad \Longleftrightarrow \qquad \begin{cases} 1) \quad \forall n \in \mathbb{N}_0 \quad \dots \\ \\ 2) \end{cases}$$

Multiplication operator: boundness and norm

Exercise 21 (definition of the supremum norm, review)**.** Let $a \in \ell^\infty$. Put

$$\nu \coloneqq \|a\|_\infty = \sup_{n \in \mathbb{N}_0} |a_n|.$$

Write the definition of ν using the quantifications $\forall$ and $\exists$ and some of the inequalities $<, >, \le, \ge$, without mentioning explicitly the concept of upper bounds.

Exercise 22 (boundness of the multiplication operator)**.** Let $a \in \ell^\infty$ and $x \in \ell^2$. Denote $\|a\|_\infty$ by ν. Prove that $\|M_a x\|_2 \le \nu \|x\|_2$.

$$\|M_a x\|_2^2 = \|\underbrace{\qquad\qquad}_{?}\|_2^2 = \sum_{j=0}^{\infty} \underbrace{\qquad\qquad\qquad}_{?} \le$$

Exercise 23. Let $a \in \ell^\infty$. Put $\nu \coloneqq \|a\|_\infty$. Given a number $\gamma < \nu$, construct a sequence $x \in \ell^2$ such that

$$\|x\|_2 = 1 \qquad \text{and} \qquad \|M_a x\|_2 \ge \gamma.$$

Exercise 24 (norm of a bounder linear operator, review)**.** Let $T\colon \ell^2 \to \ell^2$ be a bounded linear operator. Recall the definition of the norm of T:

$$\|T\| \coloneqq$$

Exercise 25 (norm of the multiplication operator)**.** Let $a \in \ell^\infty$. Calculate $\|M_a\|$.

In the previous exercises we deduced a formula for the norm of $\|M_a\|$ using the definition of supremum. There is another way to prove one part of this result.

Exercise 26 (monotonicity of the image of a subset under a function)**.** Let $f\colon X \to Y$ be an arbitrary function and $A \subset B \subset X$. Compare the images of A and B under the function f (choose $\subset$ or $\supset$):

$$f(A) \underbrace{\qquad}_{?} f(B).$$

Exercise 27. Let $a \in \ell^\infty$. Compare the following sets (put $\subset$ or $\supset$):

$$\begin{aligned}
\big\{e_n\colon\quad n \in \mathbb{N}_0\big\} &\underbrace{\qquad}_{?} \big\{x \in \ell^2\colon\quad \|x\|_2 = 1\big\} \\
\big\{M_a e_n\colon\quad n \in \mathbb{N}_0\big\} &\underbrace{\qquad}_{?} \big\{M_a x\colon\quad x \in \ell^2,\quad \|x\|_2 = 1\big\}, \\
\big\{\|M_a e_n\|_2\colon\quad n \in \mathbb{N}_0\big\} &\underbrace{\qquad}_{?} \big\{\|M_a x\|_2\colon\quad x \in \ell^2,\quad \|x\|_2 = 1\big\}.
\end{aligned}$$

Exercise 28 (monotonicity of the supremum of a set)**.** Let $A, B \in \overline{\mathbb{R}}$ such that $A \subset B$. Compare the supremums of A and B:

$$\sup(A) \underbrace{\qquad}_{?} \sup(B).$$

Exercise 29. Applying the results of two previous exercises deduce an inequality between $\|M_a\|$ and $\|a\|_\infty$.

Linear operations with multiplication operators

Exercise 30 (linear operations with bounded linear operators, review)**.** Recall the definition of the linear operations with bounded linear operators. If $T\colon \ell^2 \to \ell^2$ and $S\colon \ell^2 \to \ell^2$ are bounded linear operators and $\lambda \in \mathbb{C}$, then for all $x \in \ell^2$

$$(T+S)(x) \coloneqq \underbrace{\qquad\qquad\qquad}_{?}, \qquad\qquad (\lambda T)(x) \coloneqq \underbrace{\qquad\qquad\qquad}_{?}.$$

Exercise 31. Let $a, b \in \ell^\infty$. Calculate $M_a + M_b$.

Solution. Let x be an arbitrary element of ℓ^2. Then

$$(M_a + M_b)(x) = \qquad\qquad\qquad\qquad\qquad\qquad\qquad.$$

On the other hand,

$$M_{a+b}(x) =$$

So we see that for every $x \in \ell^2$ the following vectors are equal:

$$\underbrace{\qquad\qquad}_{?} = \underbrace{\qquad\qquad}_{?}.$$

Since x is arbitrary, we conclude that the following operators are equal:

$$M_a + M_b = \underbrace{\qquad\qquad\qquad}_{?}.$$

□

Exercise 32. Let $a \in \ell^\infty$ and $\lambda \in \mathbb{C}$. Calculate λM_a.

Product of multiplication operators

Exercise 33 (product of bounded linear operators, review)**.** Recall the definition of the product of bounded linear operators. If $T\colon \ell^2 \to \ell^2$ and $S\colon \ell^2 \to \ell^2$ are bounded linear operators, then for all $x \in \ell^2$

$$(TS)(x) := \underbrace{\qquad\qquad\qquad}_{?}.$$

Exercise 34. Let $a, b \in \ell^\infty$. Calculate $M_a M_b$.

Adjoint operator of the multiplication operator

Exercise 35 (adjoint to a bounded linear operator, review)**.** Recall the definition of the adjoint to a bounded linear operator. Let $T\colon \ell^2 \to \ell^2$ is a bounded linear operator. A bounded linear operator $S\colon \ell^2 \to \ell^2$ is called *adjoint* (or *Hermitian conjugate*) of T if for all $x, y \in \ell^2$

$$\langle x, Ty \rangle = \underbrace{\qquad\qquad\qquad}_{?}.$$

It is known that there exists a unique operator S satisfying this condition. This operator is denoted by T^*.

Exercise 36. Let $a \in \ell^\infty$. Calculate M_a^*, that is, the adjoint of the operator M_a.

Solution. For all $x, y \in \ell^2$,

$$\langle x, M_a y \rangle =$$

Therefore for all $x \in \ell^2$,

$$M_a^*(x) = \underbrace{\qquad\qquad\qquad}_{?}.$$

Conclusion: $\quad M_a^* = \underbrace{\qquad\qquad}_{?}.$ □