Criteria of multiplication operator on the space of square-summable sequences

Objectives. Establish some criteria of the multiplication operator on the space ℓ^2 .

Requirements. The space ℓ^2 of the square-summable sequences of complex numbers, multiplication operator by a bounded sequence on the space ℓ^2 , basic properties of the multiplication operator.

Denote by \mathbb{N} the set of the natural numbers. The notation $\mathbb{C}^{\mathbb{N}}$ will be used for the set of all the sequences of complex numbers, that is, for the set of all the functions $\mathbb{N} \to \mathbb{C}$.

The space of the square-summable sequences (review)

Exercise 1. Recall the definition of $\ell^2(\mathbb{N})$ and the definition of the inner product in $\ell^2(\mathbb{N})$. We suppose that the inner product is linear with respect to the second argument.

$$\ell^2 \coloneqq \ell^2(\mathbb{N}) \coloneqq \bigg\{ x \in \mathbb{C}^{\mathbb{N}} \colon \sum \hspace{1cm} \big\}, \hspace{1cm} \langle x,y \rangle \coloneqq \sum$$

Exercise 2 (the space of the bounded sequences of complex numbers).

$$\ell^{\infty} \coloneqq \ell^{\infty}(\mathbb{N}) \coloneqq \bigg\{ x \in \mathbb{C}^{\mathbb{N}} \colon \bigg\}.$$

Canonical base of ℓ^2 (review)

Exercise 3 (canonical base of ℓ^2). For every $n \in \mathbb{N}$, the sequence e_n is defined by:

$$e_n \coloneqq \left(\underbrace{}_?\right)_{j \in \mathbb{N}}.$$

.

Exercise 4. Let $x \in \ell^2$ and $n \in \mathbb{N}$. Calculate: $\langle e_n, x \rangle =$

Exercise 5. It is known that $(e_n)_{n \in \mathbb{N}}$ is an orthonormal base of ℓ^2 . Therefore every $x \in \ell^2$ can be decomposed in the following manner:

$$x = \sum_{n \in \mathbb{N}} \langle , \rangle e_n = \sum_{n \in \mathbb{N}} \underbrace{}_? e_n.$$

Exercise 6. Let $b \in \ell^2$ and $k \in \mathbb{N}$ such that

$$\forall j \in \mathbb{N} \qquad \langle e_j, b \rangle = 0.$$

Find a formula for b.

Definition and basic properties of the multiplication operator (review)

Exercise 7 (point-wise product of sequences). Let $x, y \in \mathbb{C}^{\mathbb{N}}$. The *point-wise product* or *component-wise product* of the sequences x and y is defined by:

$$\forall n \in \mathbb{N}$$
 $(xy)_n \coloneqq \underbrace{}_?, \quad \text{or} \quad xy \coloneqq (\underbrace{}_?)_{n \in \mathbb{N}}.$

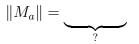
Definition 1 (definition of the multiplication operator). Let $a \in \ell^{\infty}$. The multiplication operator by the sequence a is an operator $\ell^2 \to \ell^2$ defined by:

$$\forall x \in \ell^2 \qquad M_a x \coloneqq \underbrace{\qquad}_?$$

$$\forall x \in \ell^2 \qquad \forall j \in \mathbb{N} \qquad (M_a x)(j) \coloneqq \underbrace{\qquad}_?$$

that is,

Exercise 8 (norm of the multiplication operator). Let $a \in \ell^{\infty}$. Then M_a is bounded and



Exercise 9 (algebraic operations with multiplication operators, review). Let $a, b \in \ell^{\infty}$ and $\lambda \in \mathbb{C}$. Recall the formulas:

$$M_a + M_b = \lambda M_a =$$

 $(M_a)^* = M_a M_b =$

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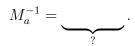
Invertibility and spectrum of the multiplication operator (review)

Definition 2. Let $a \in \ell^{\infty}$. Denote by $C\mathcal{R}(a)$ the closure of the range of a.

Exercise 10 (criterion of invertibility of the multiplication operator). Let $a \in \ell^{\infty}$. Then the following three conditions are equivalent:

 $M_a \text{ is invertible} \iff \inf_{n \in \mathbb{N}} |a_n| \underbrace{\qquad}_? \iff 0 \notin \underbrace{\qquad}_?.$

If these conditions are fulfilled, then



Exercise 11 (spectrum of the multiplication operator). Let $a \in \ell^{\infty}$. Then

$$\operatorname{Sp}(M_a) = \underbrace{}_{?}$$
.

Multiplication operator and canonical base

Exercise 12. Let $a \in \ell^{\infty}$, $x \in \ell^2$ and $n \in \mathbb{N}$. Calculate:

$$\langle e_n, M_a x \rangle =$$

Exercise 13. Let $a \in \ell^{\infty}$ and $x \in \ell^2$. Prove that

$$M_a x = \sum_{j \in \mathbb{N}} a_j \langle e_j, x \rangle \, e_j.$$

Exercise 14. Let $a \in \ell^{\infty}$ and $j \in \mathbb{N}$. Calculate:

$$\langle e_j, M_a e_j \rangle =$$

Exercise 15. Let $a \in \ell^{\infty}$ and the sequence $(\lambda_j)_{j \in \mathbb{N}}$ is defined by:

$$\forall j \in \mathbb{N} \qquad \lambda_j \coloneqq \langle e_j, M_a e_j \rangle.$$

Express a through λ_j .

Exercise 16. Let $a \in \ell^{\infty}$ and $j, k \in \mathbb{N}$ such that $j \neq k$. Calculate:

$$\langle e_j, M_a e_k \rangle =$$

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Criteria of multiplication operator on ℓ^2

Exercise 17. Let $k \in \mathbb{N}$ and $A: \ell^2 \to \ell^2$ be a bounded linear operator such that

$$\forall j \in \mathbb{N} \setminus \{k\} \qquad \langle e_j, Ae_k \rangle = 0.$$

Find a formula for Ae_k .

Exercise 18. Let $A: \ell^2 \to \ell^2$ be a bounded linear operator. Prove that the following conditions are equivalent:

- (a) There exists $a \in \ell^{\infty}$ such that $A = M_a$.
- (b) There exists $a \in \ell^{\infty}$ such that for all $x \in \ell^2$,

$$Ax = \sum_{j \in \mathbb{N}} a_j \langle e_j, x \rangle \, e_j.$$

(c) For all $j, k \in \mathbb{N}$ with $j \neq k$,

$$\langle e_j, Ae_k \rangle = 0.$$