Continuous prolongation of a uniformly continuous function defined on a dense subset of a metric space

Objectives. Prove that a uniformly continuous function defined on a dense subset of a metric space has a unique continuous prolongation to the whole metric space.

Requirements. Metric space, convergence of sequences, subsequences, convergence of subsequences, Cauchy sequence, dense subset, modulus of continuity, uniformly continuous function.

In all exercises we suppose that (X, ρ) is a metric space.
Denote by $\mathbb{N}$ the set $\{0, 1, 2, \ldots\}$.

Convergent sequences (review)

Exercise 1 (definition of convergent sequence)**.** Let $x \in X$ and $(y_n)_{n\in\mathbb{N}}$ be a sequence. Complete the definition:

$$\lim_{n\to\infty} y_n = x \qquad \overset{\text{def}}{\iff} \qquad \forall \varepsilon > 0$$

Exercise 2 (convergence of a sequence implies convergence of the subsequences)**.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and $(\nu(k))_{k\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$. Define $(y_k)_{k\in\mathbb{N}}$ by $y_k \coloneqq x_{\nu(k)}$. Suppose that $(x_n)_{n\in\mathbb{N}}$ converges to a point $z \in X$. Prove that $(y_k)_{k\in\mathbb{N}}$ also converges to z.

Exercise 3 (merge two sequences converging to the same point)**.** Let $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ be two sequences in X converging to the same point $x \in X$:

$$\lim_{n\to\infty} y_n = x, \qquad \lim_{n\to\infty} z_n = x.$$

Prove that the sequence $(t_n)_{n\in\mathbb{N}}$ defined in the following manner also converges to x:

$$t_n := \begin{cases} y_k, & \text{if } n = 2k, \quad k \in \mathbb{N}; \\ z_k, & \text{if } n = 2k+1, \quad k \in \mathbb{N}. \end{cases}$$

Convergent sequences and dense subsets (review)

Exercise 4 (definition of the closure in terms of sequences)**.** Let Y be a subset of X.

$$x \in \overline{Y} \quad \iff$$

Exercise 5 (definition of dense subset in terms of sequences)**.** A subset Y of X is called *dense* if

$$\forall x \in$$

Cauchy sequences and complete spaces (review)

Exercise 6. Recall the definition: a sequence $(x_n)_{n\in\mathbb{N}}$ is called a *Cauchy sequence* if

$$\forall \varepsilon > 0$$

Exercise 7. Let (X, ρ) be a metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence. Consider the following conditions:

(a) $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

(b) $(x_n)_{n\in\mathbb{N}}$ converges, that is, there exists a point $p \in X$ such that $\lim\limits_{n\to\infty} x_n = p$.

What logical relation between (a) and (b) is always true? $\Rightarrow$ or $\Leftarrow$?

$$\text{(a)} \quad \underbrace{\qquad\qquad}_{?} \quad \text{(b)}.$$

Exercise 8. Recall the definition: a metric space (X, ρ) is called *complete* if ...

Exercise 9. Recall which of the following metric spaces are complete:

$$\mathbb{R}, \qquad \mathbb{Q}, \qquad \mathbb{C}.$$

Uniformly continuous functions (review)

In the following exercises we suppose that (X,ρ) is a metric space.

Exercise 10 (definition of the modulus of continuity of a function)**.** Let $f\colon X \to \mathbb{C}$ and $\delta > 0$. Recall the definition:

$$\omega_{\rho,f}(\delta) \coloneqq$$

Exercise 11 (definition of uniformly continuous function in terms of its modulus of continuity)**.** A function $f\colon X \to \mathbb{C}$ is called *uniformly continuous* if

$$\omega_{\rho,f}(\delta)$$

Exercise 12 (definition of uniformly continuous function in terms of ε and δ)**.** A function $f\colon X \to \mathbb{C}$ is called *uniformly continuous* if

$$\forall \varepsilon > 0$$

Exercise 13 (values of a uniformly continuous function on a Cauchy sequence)**.** Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X and $f\colon X \to \mathbb{C}$ be a uniformly continuous function. Prove that $\bigl(f(x_n)\bigr)_{n\in\mathbb{N}}$ is a Cauchy sequence. Does it converges?

Construction of a continous prolongation of a uniformly continuous function defined on a dense subset of a metric space

In the exercises of this section we suppose that Y is a dense subset of X and $f\colon Y \to \mathbb{C}$ is uniformly continuous on Y with respect to the distance ρ:

$$\lim_{\delta\to 0} \sup\big\{|f(y') - f(y'')|\colon \quad y', y'' \in Y, \quad \rho(y', y'') \le \delta\big\} = 0.$$

Exercise 14 (the sequence of the values of f on a Cauchy sequence is a Cauchy sequence)**.** Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in Y. Prove that $\big(f(y_n)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Exercise 15 (the sequence of the values of f on a converging sequence has a limit)**.** Let $x \in X$ and $(y_n)_{n\in\mathbb{N}}$ be a sequence in Y converging to x:

$$\Big(\forall n \in \mathbb{N} \quad y_n \in Y \Big) \qquad \wedge \qquad \Big(\lim_{n\to\infty} y_n = x \Big).$$

Prove that the sequence $\big(f(y_n)\big)_{n\in\mathbb{N}}$ has a limit.

Exercise 16 (the sequences of the values of f on two sequences converging to the same point have the same limit)**.** Let $x \in X$. Let $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ be two sequences in Y converging to x. Prove that

$$\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(z_n).$$

Now we are ready to construct a continuous prolongation of f.

Exercise 17. Define $g\colon X \to \mathbb{C}$ by the following rule: given a point $x \in X$, put

$$g(x) \coloneqq \lim_{n\to\infty} f(y_n),$$

where $(y_n)_{n\in\mathbb{N}}$ is a sequence converging to x.

1. Explain why such a sequence $(y_n)_{n\in\mathbb{N}}$ exists.
2. Explain why the limit does not depend on the choice of $(y_n)_{n\in\mathbb{N}}$.

Exercise 18 (g is a prolongation of f)**.** Prove that $g(y) = f(y)$ for all $y \in Y$.

Here we suppose that g is a prolongation of f constructed in the Exercise 17.

Exercise 19. Prove that

$$\begin{aligned}\sup\bigl\{|f(y_1)-f(y_2)|:\quad y_1,y_2\in Y,\quad \rho(y_1,y_2)\le\delta\bigr\}&\le\\ \le\sup\bigl\{|g(x_1)-g(x_2)|:\quad x_1,x_2\in X,\quad \rho(x_1,x_2)\le\delta\bigr\}&.\end{aligned}$$

Exercise 20 (hard)**.** Prove that for all $\delta>0$

$$\begin{aligned}\sup\bigl\{|g(x_1)-g(x_2)|:\quad x_1,x_2\in X,\quad \rho(x_1,x_2)\le\delta\bigr\}&\le\\ \le\sup\bigl\{|f(y_1)-f(y_2)|:\quad y_1,y_2\in Y,\quad \rho(y_1,y_2)\le 3\delta\bigr\}&.\end{aligned}$$

Exercise 21. Prove that g is uniformly continuous.

Exercise 22 (uniqueness of the continuous prolongation)**.** Let Y be a dense subset of a metric space and $g, h \in C(X, \mathbb{C})$ be some continuous functions such that

$$\forall y \in Y \qquad g(y) = h(y).$$

Prove that $g(x) = h(x)$ for all $x \in X$.