Bounded uniformly continuous functions

Objectives. To study the basic properties of the C^*-algebra of the bounded uniformly continuous functions on some metric space.

Requirements. Basic concepts of analysis: supremum, limit, continuity.

Exercise 1 (definition of metric)**.** Let X be a set and $\rho\colon X \times X \to \mathbb{R}$ be a function. The function ρ is called *metric* (or *distance*) if the following conditions hold:

1.
2.
3. For all $x \in X, \quad \rho(x, x) = 0.$
4. For all $x, y \in X$ such that $x \neq y, \quad \rho(x, y) > 0.$

Modulus of continuity of a function

Denote by $\mathbb{R}_+$ the set $(0, +\infty)$.

Definition 1 (modulus of continuity of a function)**.** Let $f\colon X \to \mathbb{C}$. Define the function $\omega_{\rho,f}\colon \mathbb{R}_+ \to [0, +\infty]$ by the following rule:

$$\omega_{\rho,f}(\delta) \coloneqq \sup\bigl\{|f(x) - f(y)|: \quad \rho(x, y) \le \delta\bigr\}.$$

Exercise 2 (example)**.** Consider the set $[0, +\infty)$ with the usual metric $d(x, y) \coloneqq |x - y|$. Define $f\colon [0, +\infty) \to \mathbb{C}$ by

$$f(x) \coloneqq \sqrt{x}.$$

For all $\delta > 0$ compute $\omega_{d,f}(\delta)$.

Monotonicity of the modulus of continuity

Notation 1 (subset of a set)**.** Let A, B be some sets. We write $A \subset B$ iff A is a subset of B, that is, iff every element of A belongs to B. Note that for every set A the conclusion $A \subset A$ holds.

Definition 2 (types of monotone functions)**.** Let $h\colon D \to [-\infty, +\infty]$ where $D \subset [-\infty, +\infty]$. The function h is called:

- *increasing* iff $\qquad \forall x, y \in D \qquad \big(x < y \qquad \Longrightarrow \qquad f(x) \le f(y)\big);$
- *decreasing* iff $\qquad \forall x, y \in D \qquad \big(x < y \qquad \Longrightarrow \qquad f(x) \ge f(y)\big);$
- *strictly increasing* iff $\qquad \forall x, y \in D \qquad \big(x < y \qquad \Longrightarrow \qquad f(x) < f(y)\big);$
- *strictly decreasing* iff $\qquad \forall x, y \in D \qquad \big(x < y \qquad \Longrightarrow \qquad f(x) > f(y)\big).$

Exercise 3. Let $f\colon X \to \mathbb{C}$, $\delta_1, \delta_2 > 0$, $\delta_1 < \delta_2$.

1. Compare the following subsets of $X \times X$:

$$\big\{(x, y) \in X^2\colon \quad \rho(x, y) \le \delta_1\big\} \qquad \underbrace{\qquad\qquad}_{\subset \text{ or } \supset} \qquad \big\{(x, y) \in X^2\colon \quad \rho(x, y) \le \delta_2\big\}.$$

2. Compare the following subsets of $[0, +\infty]$:

$$\big\{|f(x) - f(y)|\colon \quad \rho(x, y) \le \delta_1\big\} \qquad \underbrace{\qquad\qquad}_{\subset \text{ or } \supset} \qquad \big\{|f(x) - f(y)|\colon \quad \rho(x, y) \le \delta_2\big\}.$$

3. Compare the following numbers:

$$\omega_{\rho,f}(\delta_1) \qquad \underbrace{\qquad\qquad}_{\le \text{ or } \ge} \qquad \omega_{\rho,f}(\delta_2).$$

Exercise 4. Let $f\colon X \to \mathbb{C}$. From to the previous exercise make a conclusion about the monotonicity of $\omega_{\rho,f}$.

Supremum-norm and bounded functions

Definition 3 (supremum-norm)**.** Given a function $f\colon X \to \mathbb{C}$ denote by $\|f\|_\infty$ its supremum-norm:

$$\|f\|_\infty \coloneqq \sup_{x\in X} |f(x)|.$$

Exercise 5. Let $f, g\colon X \to \mathbb{C}$ be bounded functions and $\lambda \in \mathbb{C}$. Recall exact formulas or upper bounds for the supremum-norm of the functions $f+g$, λf, $\overline{f}$ and fg (just write formulas without any proofs):

$\|f+g\|_\infty \leq$ $\qquad\qquad$ $\|\lambda f\|_\infty =$

$\|\overline{f}\|_\infty$ $\qquad\qquad$ $\|fg\|_\infty$

Exercise 6. Prove that the space of the bounded functions $X \to \mathbb{C}$ is complete (with respect to the distance induced by the supremum-norm).

Arithmetic properties of the modulus of continuity

Exercise 7. Let $f, g\colon X \to \mathbb{C}$ be bounded functions. Find an upper bound of $\omega_{\rho,f+g}$ in terms of $\omega_{\rho,f}$ and $\omega_{\rho,g}$.

Exercise 8. Let $f\colon X \to \mathbb{C}$ and $\lambda \in \mathbb{C}$. Express $\omega_{\rho,\lambda f}$ through $\omega_{\rho,f}$ and λ.

Exercise 9. Let $f\colon X \to \mathbb{C}$. Express $\omega_{\rho,\overline{f}}$ through $\omega_{\rho,f}$.

Exercise 10. Let $f, g\colon X \to \mathbb{C}$ be bounded functions. Find an upper bound of $\omega_{\rho,fg}$ in terms of $\omega_{\rho,f}$, $\omega_{\rho,g}$, $\|f\|_\infty$ and $\|g\|_\infty$.

Uniform approximation and modulus of continuity

Exercise 11. Let $f, g\colon X \to \mathbb{C}$ and $x, y \in X$. Find an upper bound of $|f(x) - f(y)|$ in terms of $|g(x) - g(y)|$ and $\|f - g\|_\infty$.

$$\begin{aligned} |f(x) - f(y)| &= \big|f(x) - \underbrace{\qquad}_{?} + \underbrace{\qquad}_{?} - \underbrace{\qquad}_{?} + \underbrace{\qquad}_{?} - f(y)\big| \\ &\le \end{aligned}$$

Exercise 12. Let $f, g\colon X \to \mathbb{C}$. Find an upper bound of $\omega_{\rho,f}$ in terms of $\omega_{\rho,g}$ and $\|f - g\|_\infty$:

$$\omega_{\rho,f}(\delta) \le$$

Uniformly continuous functions

Definition 4 (uniformly continuous function)**.** A function $f\colon X \to \mathbb{C}$ is called *uniformly continuous* if

$$\lim_{\delta \to 0} \omega_{\rho,f}(\delta) = 0.$$

Exercise 13. Write the previous definition using quantifiers $\forall$ and $\exists$:

$$\lim_{\delta \to 0} \omega_{\rho,f}(\delta) = 0 \qquad \overset{\text{def}}{\Longleftrightarrow} \qquad \forall \varepsilon > 0$$

Exercise 14. Simplify the previous definition using the fact that $\omega_{\rho,f}$ is monotone:

$$\lim_{\delta \to 0} \omega_{\rho,f}(\delta) = 0 \qquad \Longleftrightarrow \qquad \forall \varepsilon > 0$$

Lipschitz-continuous functions

Exercise 15. Write the definition of Lipschitz-continuous function in terms of $\omega_{\rho,f}$.

Exercise 16. Prove that every Lipschitz-continuous function is uniformly continuous.

Bounded uniformly continuous functions

Definition 5. Denote by $\mathrm{UC_b}(X, \rho, \mathbb{C})$ the set of the bounded uniformly continuous functions, with usual arithmetic operations, point-wise conjugation and supremum-norm. Breve notation: $\mathrm{UC_b}(X)$.

Exercise 17. Prove that every uniformly continuous function $f\colon X \to \mathbb{C}$ is continuous.

Exercise 18. Prove that $\mathrm{UC_b}(X)$ is closed under arithmetic operations and point-wise conjugation.

Exercise 19. Prove that $\mathrm{UC_b}(X)$ is a closed subspace of the space of the bounded functions $X \to \mathbb{C}$.

Exercise 20. Recall the definition of C^*-algebra. Determine whether $\mathrm{UC_b}(X)$ is a C^*-algebra or not.