Another special approximate unit

Objectives. To study some basic properties of the functions ϕ_n and their Laplace transforms ψ_n. Here ϕ_n is defined on $\mathbb{R}_+$ by

$$\phi_n(v) \coloneqq \frac{1}{n!\,(n-2)!}\,\frac{d^n}{dv^n}\left(\mathrm{e}^{-v}\ v^{2n-1}\right).$$

Requirements. Gamma function and its basic properties, beta function and its basic properties, expression of the beta function through the gamma function, basic integration tecnics (change of variables and integration by parts), general Leibniz rule, Laplace transform.

Definition of ϕ_n

Denote $(0,+\infty)$ by $\mathbb{R}_+$.

Definition 1. For all $n \in \{1,2,\ldots\}$ define the function ϕ_n on $\mathbb{R}_+$ by

$$\phi_n(v) \coloneqq \frac{1}{n!\,(n-2)!}\,\frac{d^n}{dv^n}\left(\mathrm{e}^{-v}\ v^{2n-1}\right). \tag{1}$$

Exercise 1. Calculate the first derivative of $\mathrm{e}^{-v}\ v^n$:

$$\frac{d}{dv}\left(\mathrm{e}^{-v}\ v^n\right) =$$

Exercise 2. Calculate $\phi_2(v)$.

Exercise 3. Calculate $\phi_3(v)$.

Exercise 4. Recall the Binomial Formula:

$$(a+b)^n = \sum$$

Exercise 5. Calculate the first three derivatives of the product fg. Here f and g are functions of one variable denoted by v.

$$\frac{d}{dv}(fg) = f'g + fg',$$

$$\frac{d^2}{dv^2}(fg) =$$

$$\frac{d^3}{dv^3}(fg) =$$

Exercise 6. Write the formula (*general Leibniz rule*) for the k-st derivative of a product:

$$\frac{d^k}{dv^k}(fg) = \sum$$

Exercise 7. Calculate the k-st derivative of a power v^n, $k \leq n$. Use factorials or the gamma function to express the coefficient.

$$\frac{d^k}{dv^k}\left(v^n\right) =$$

Exercise 8. Recall the definition (1) of the function ϕ_n:

$$\phi_n(v) =$$

Exercise 9. Express the function ϕ_n as a certain sum (free of derivatives):

$$\phi_n(v) = \qquad \sum \tag{2}$$

Exercise 10. The previous exercise permit us to make some conclusions the form of the function ϕ_n. The function ϕ_n can be written as a product

$$\underbrace{\qquad\qquad}_{?} Q_n(v)$$

where Q_n is a polynomial,

$$\deg(Q_n) = \underbrace{\qquad\qquad}_{?},$$

and the coefficients of x^k for $k < \underbrace{\qquad\qquad}_{?}$ are equal to zero.

Exercise 11. Formally we define the functions ϕ_n only on $\mathbb{R}_+ = (0, +\infty)$, but they can be extended continuously onto $(-\infty, +\infty]$. In particular, we are interested in the (limit) values of ϕ_n at the points 0 and $+\infty$:

$$\lim_{v \to 0^+} \phi_n(v) =$$

$$\lim_{v \to +\infty} \phi_n(v) =$$

Functions ϕ_n and associated Laguerre polynomials

Exercise 12. Find the definition for the **associated Laguerre polynomials** (= generalized Laguerre polynomials = Sonine polynomials) in the following form (the Rodrigues representation):

$$L_m^k(v) = \qquad \frac{d^m}{dv^m}\left(\mathrm{e}^{-v} \qquad \right). \tag{3}$$

Find also the explicit formula for the associated Laguerre polynomials:

$$L_m^k(v) = \sum \tag{4}$$

Exercise 13. Comparing the definition (1) of ϕ_n with (3) express ϕ_n through some associated Laguerre polynomial.

$$\phi_n(v) = \tag{5}$$

Exercise 14. Substitute (4) into (5) and compare the obtained result with the formula (2) from Exercise 9.

Some basic properties of the gamma function (review)

Exercise 15. Recall the definition of the gamma function:

$$\Gamma(x) := \int_0^{+\infty} \underbrace{\qquad\qquad\qquad}_{?} \, dt.$$

Exercise 16. Integrating by parts express $\Gamma(x+1)$ through $\Gamma(x)$:

$$\Gamma(x+1) =$$

Exercise 17. Compute $\Gamma(1)$:

$$\Gamma(1) = \int_0^{+\infty}$$

Exercise 18. Compute $\Gamma(2)$, $\Gamma(3)$, $\Gamma(4)$, $\Gamma(5)$:

$$\Gamma(2) = \underbrace{\qquad}_{?} \Gamma(1) = \qquad\qquad \Gamma(3) = \underbrace{\qquad}_{?} \Gamma(2) =$$

$$\Gamma(4) = \qquad\qquad \Gamma(5) =$$

Exercise 19. Express $\Gamma(n)$ through the factorial function for $n \in \{1, 2, 3, \ldots\}$.

$$\Gamma(n) =$$

Exercise 20. Let $a > 0$ and $p > 0$. Express the following integral through the gamma function (make a suitable change of variables):

$$\int_0^{+\infty} x^p \, \mathrm{e}^{-ax} \, dx =$$

Some basic propierties of the beta function (review)

Exercise 21. Recall the definition of the beta function:

$$\mathrm{B}(x,y) := \int_0^1 \underbrace{\qquad\qquad\qquad\qquad}_{?} \, du.$$

Exercise 22. Express the following integral through the beta function:

$$\int_0^1 u^\alpha (1-u)^\beta \, du = \mathrm{B}(\underbrace{\qquad\qquad\qquad}_{?}).$$

Exercise 23. Recall the formula that expresses the beta function through the gamma function:

$$\mathrm{B}(x,y) = \frac{\qquad\qquad\qquad}{\qquad\qquad\qquad}.$$

Exercise 24. Let $p, q \in \{1, 2, 3, \ldots\}$. Using the formula from the previous exercise express $\mathrm{B}(p,q)$ through some factorials.

$$\mathrm{B}(p,q) =$$

Exercise 25. Prove that the function $t \mapsto \dfrac{t}{t+1}$ is strictly increasing on $\mathbb{R}_+$.

Exercise 26. Calculate the limits:

$$\lim_{t \to 0} \frac{t}{t+1} = \qquad\qquad \lim_{t \to +\infty} \frac{t}{t+1} =$$

Exercise 27. Find the range of the function defined on $\mathbb{R}_+$ by $t \mapsto \frac{t}{t+1}$.

Exercise 28. Let $u = \dfrac{t}{t+1}$ where $t > 0$. Express t through u.

Exercise 29. Making a suitable change of variables express the following integral through the beta function:

$$\int_0^{+\infty} \frac{t^a\,dt}{(1+t)^b} =$$

ψ_n := the Laplace transform of ϕ_n

Recall the definition of ϕ_n:

$$\phi_n(v) \coloneqq \frac{1}{n!\,(n-2)!}\,\frac{d^n}{dv^n}\left(\mathrm{e}^{-v}\ v^{2n-1}\right).$$

Definition 2. For each $n \in \{1, 2, \ldots\}$ define the function $\psi_n \colon \mathbb{R}_+ \to \mathbb{R}$ as the Laplace transform of the function ϕ_n:

$$\psi_n(t) \coloneqq \int\limits_0^{+\infty} \phi_n(v)\ \mathrm{e}^{-vt}\ dv. \tag{6}$$

Exercise 30. Calculate ψ_1.

Exercise 31. Calculate ψ_2.

Exercise 32. Calculate ψ_3.

Exercise 33. Using the definition (1) of ϕ_n calculate ψ_n. The coefficient in the answer can be written through the beta function.

Some properties of ψ_n

Exercise 34. Let $n \in \{2,3,\ldots\}$. Calculate the integral of ψ_n on $\mathbb{R}_+$:

$$\int_0^{+\infty} \psi_n(t)\,dt =$$

Exercise 35. To verify the result of the previous exercise, calculate the following integral:

$$\int_0^{+\infty} \psi_2(t)\,dt =$$

Exercise 36. Let $\delta > 0$. Prove that

$$\lim_{n\to\infty} \sup_{0<t\le \mathrm{e}^{-\delta}} \psi_n(t) = 0.$$

Exercise 37. Let $\delta > 0$. Calculate the limit:

$$\lim_{n\to\infty} \int_0^{\mathrm{e}^{-\delta}} \psi_n(t)\,dt =$$

Exercise 38. Let $\delta > 0$. Make the change of variables $s = \dfrac{1}{t}$ in the following integral:

$$\int_{\mathrm{e}^{\delta}}^{+\infty} \psi_n(t)\,dt =$$

Exercise 39. Let $\delta > 0$. Calculate the limit:

$$\lim_{n\to\infty} \int_{\left(0,\mathrm{e}^{-\delta}\right)\cup\left(\mathrm{e}^{\delta},+\infty\right)} \psi_n(t)\,dt =$$