Analytic functions of two variables vanishing on the conjugate diagonal

This text is a rough draft.

These exercises are based on the article of Karel Stroethoff, The Berezin transform and operators on spaces of analytic functions.

Objectives. Let Ω be an open subset of the complex plane $\mathbb{C}$. Denote by Ω^* the set

$$\Omega^* \coloneqq \big\{ w \in \mathbb{C} \colon \ \overline{w} \in \Omega \big\}.$$

Suppose that $h\colon \Omega \times \Omega^* \to \mathbb{C}$ is analytic and $h(z, \bar{z}) = 0$ for all $z \in \Omega$. Prove that $h = 0$. This result is used by Karel Stroethoff to prove that the Berezin transform of bounded linear operators is an injective map.

Requirements. Analytic functions of real variable, analytic functions of complex variable, analytic functions of two complex variables, linear independent functions.

Linear independency of the exponent functions

Exercise 1. Recall the definition of the *Vandermonde matrix* generated by some numbers $a_1, \ldots, a_n \in \mathbb{C}$ and the formula for its determinant.

$$V(a_1, \ldots, a_n) \coloneqq$$

$$\det V(a_1, \ldots, a_n) =$$

Exercise 2. Denote by $C^k(\mathbb{R}, \mathbb{C})$ the set of the k-times continuously differentiable functions $\mathbb{R} \to \mathbb{C}$.

Exercise 3. Let $f_1, \ldots, f_n \in C^{n-1}(\mathbb{R}, \mathbb{C})$ and $x \in \mathbb{R}$. Recall the definition of the *Wronskian* of $f_1, \ldots, f_n$ at the point x:

$$W(f_1, \ldots, f_n)(x) \coloneqq$$

Exercise 4. Let $f_1, \ldots, f_n \in C^{n-1}(\mathbb{R}, \mathbb{C})$ be linearly dependent. Prove that for all $x \in \mathbb{R}$

$$W(f_1, \ldots, f_n)(x) = 0.$$

Exercise 5. Let $f_1, \ldots, f_n \in C^{n-1}(\mathbb{R}, \mathbb{C})$ and $x_0 \in \mathbb{R}$ such that

$$W(f_1, \ldots, f_n)(x_0) \neq 0.$$

Prove that $f_1, \ldots, f_n$ are linealy independent.

Exercise 6. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be pairwise different. Define the functions $f_j \colon \mathbb{R} \to \mathbb{C}$, $j \in \{1, \ldots, n\}$, by

$$f_j(x) \coloneqq \mathrm{e}^{\alpha_j x}.$$

Prove that $f_1, \ldots, f_n$ are linearly independent.

Homogeneous polynomials of two variables vanishing on the conjugate diagonal

In this section we consider a m-homogeneous polynomial of two complex variables, that is, a function $P\colon \mathbb{C}^2 \to \mathbb{C}$ of the form

$$P(z,w) \coloneqq \sum_{k=0}^{m} a_k z^k w^{m-k}.$$

Exercise 7. Let $r>0$. Suppose that $P(z,\bar{z})=0$ for all $z\in\mathbb{C}$ with $|z|=r$. Prove that for all $\theta\in\mathbb{R}$

$$\sum_{k=0}^{m} a_k\,\mathrm{e}^{2\,\mathrm{i}\,k\theta}=0.$$

Exercise 8. Let $r>0$ and $P(z,\bar{z})=0$ for all $z\in\mathbb{C}$ with $|z|=r$. Prove that

$$a_0=\ldots=a_m=0.$$

General case

Exercise 9. Let z_0 be an interior point of Ω and $r > 0$ be such a radius that

$$\{z \in \mathbb{C}\colon\ |z - z_0| < r\} \subseteq \Omega.$$

Explain why h can be written as

$$h(z, w) = \sum_{m=0}^{\infty} P_m(z - z_0, w - w_0),$$

where for every $m \in \{0, 1, 2, \ldots\}$ the function P_m is homogeneous polynomial of two variables of degree m.

Exercise 10. Let $z \in \mathbb{C}$ with $|z| < r$. Define $g\colon (-1, 1) \to \mathbb{C}$ by

$$g(t) \coloneqq h(z_0 + tz, z_0 + t\bar{z}).$$

Calculate $g(t)$ and prove that g is analytic.

Exercise 11. Suppose that $h(z, \bar{z}) = 0$ for all $z \in \Omega$. Prove that $P_m(z, \bar{z}) = 0$ for all $z \in \mathbb{C}$ with $|z| < r$.

Exercise 12. Suppose that $h(z, \bar{z}) = 0$ for all $z \in \Omega$. Prove that $h = 0$.