Approximately invertible elements in non-unital commutative Banach algebras

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joint results with
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It also appears in other areas of mathematics.

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 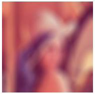 

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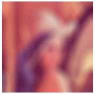

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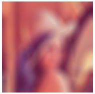

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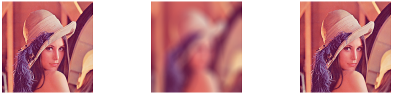

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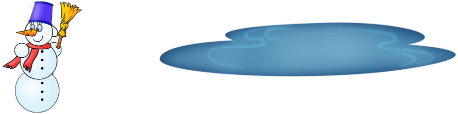

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then we can study approximately invertible elements.

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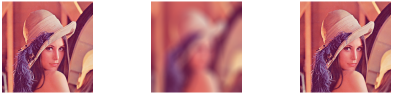

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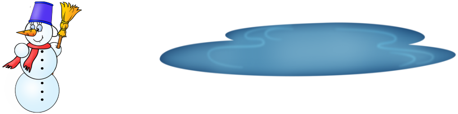

If there are approximate identities,
then we can study approximately invertible elements.

Outline

Definitions

Examples of non-unital algebras

$C_0(\mathbb{R})$

the good

$A_0(\mathbb{D})$

the bad

$L^1(\mathbb{R})$

the nice

$\mathrm{K}(\mathcal{H})$

the two-faced

General results

Banach algebras (review)

Let $\mathcal{A}$ be a normed complex vector space and at the same time an algebra. It is said that $\mathcal{A}$ is a normed algebra if the norm in $\mathcal{A}$ is *submultiplicative*:

$$\forall a, b \in \mathcal{A} \qquad \|ab\| \leq \|a\| \, \|b\|.$$

A normed algebra $\mathcal{A}$ is called a Banach algebra if $\mathcal{A}$ is *complete* with respect to the distance induced by the norm.

An algebra $\mathcal{A}$ is called unital if there exists e in $\mathcal{A}$ such that $ae = a$ for each a in $\mathcal{A}$.

Banach algebras (review)

Main examples of unital Banach algebras:

- Algebra $C_b(T, \mathbb{C})$ of bounded continuous functions $T \to \mathbb{C}$, where T is a topological space.
- Algebra $B(X, X)$ of bounded linear operators acting in a Banach space X.

For simplicity, in this talk we deal with commutative Banach algebras.

Gelfand transform (review)

Let $\mathcal{A}$ be a commutative Banach algebra.
A character (or multiplicative functional) of $\mathcal{A}$
is a non-zero algebra homomorfism $\mathcal{A} \to \mathbb{C}$.

Denote by $\mathcal{M}_{\mathcal{A}}$ the set of the characters of $\mathcal{A}$.
$\mathcal{M}_{\mathcal{A}}$ is provided with the weak-* topology.
$\mathcal{M}_{\mathcal{A}}$ is a locally compact Hausdorff space.

For each a in $\mathcal{A}$, its Gelfand transform $\widehat{a}\colon \mathcal{M}_{\mathcal{A}} \to \mathbb{C}$ is defined by

$$\widehat{a}(\varphi) = \varphi(a).$$

It is known that $\widehat{a} \in C_0(\mathcal{M}_{\mathcal{A}})$.

Put $\Gamma\colon \mathcal{A} \to C_0(\mathcal{M}_{\mathcal{A}})$, $\Gamma(a) = \widehat{a}$.
Then Γ is a homomorfism of Banach algebras, and $\|\widehat{a}\| \le \|a\|$.

Nets and convergence (review)

The concept of *nets* generalizes the concept of *sequences*.

Let J be a set and $\succeq$ be a partial order on J.
It is said that $(J, \succeq)$ is a directed set if

$$\forall p, q \in J \qquad \exists r \in J \qquad (r \succeq p) \quad \wedge \quad (r \succeq q).$$

A net in a topological space (X, τ) is a function $s\colon J \to X$, there $(J, \preceq)$ is a directed set. We write $(s_j)_{j \in J}$ instead of s.

Let (X, τ) be a topological space, $(s_j)_{j \in J}$ be a net and $y \in X$.
It is said that the net $(s_j)_{j \in J}$ converges to y iff

$$\forall V \in \tau_y \qquad \exists k \in J \qquad \forall j \succeq k \qquad s_j \in V.$$

Main definitions

Let $\mathcal{A}$ be a commutative Banach algebra.

Definition (approximate identity)

A net $(e_j)_{j\in J}$ in $\mathcal{A}$ is called an approximate identity in $\mathcal{A}$
if for every a in $\mathcal{A}$

$$\lim_{j\in J} ae_j = a.$$

Notation: $(e_j)_{j\in J} \in \mathrm{ApproxId}(\mathcal{A})$.

Definition (approximately invertible elements)

Let $x \in \mathcal{A}$. We say that x is approximately invertible
if there exists a net $(u_j)_{j\in J}$ such that $(xu_j)_{j\in J} \in \mathrm{ApproxId}(\mathcal{A})$.
Notation: $x \in \mathrm{ApproxInv}(\mathcal{A})$.

Situation in unital Banach algebras (Theorem 1)

Let $x \in \mathcal{A}$, where $\mathcal{A}$ is a unital Banach algebra with identity e.

$$
\begin{tikzcd}
\boxed{\exists y \in \mathcal{A} \quad xy = e} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \boxed{\exists (y_j)_{j\in J} \quad \lim\limits_{j\in J} xy_j = e} \arrow[d, leftrightarrow] \\
\boxed{x \in \mathrm{ApproxId}(\mathcal{A})} \arrow[r, leftrightarrow] & \boxed{\mathrm{clos}(x\mathcal{A}) = \mathcal{A}}
\end{tikzcd}
$$

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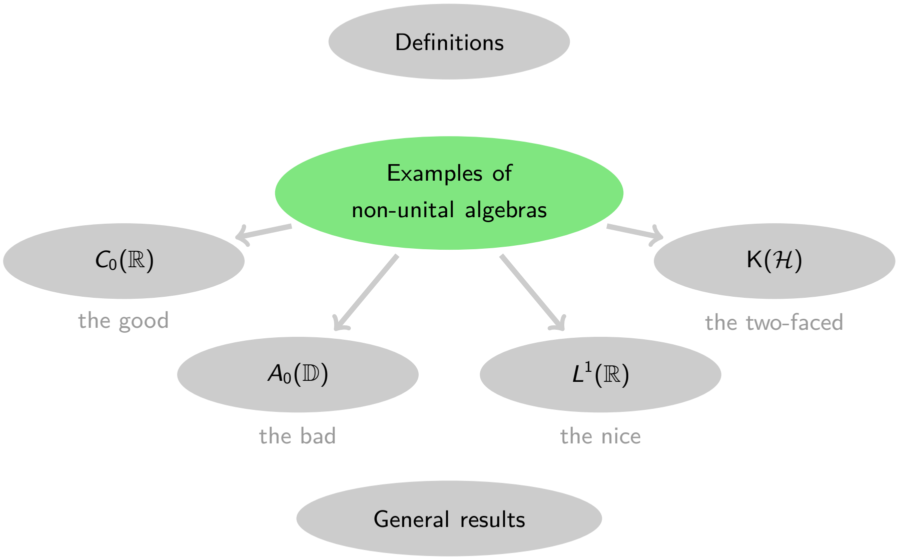
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Approximate identities in $C_0(\mathbb{R})$

$C_0(\mathbb{R}) :=$ the continuous functions on $\mathbb{R}$ that tend to zero at ∞.
$\mathcal{K} :=$ the compact subsets of $\mathbb{R}$.

Criterion of approximate identity in $C_0(\mathbb{R})$:

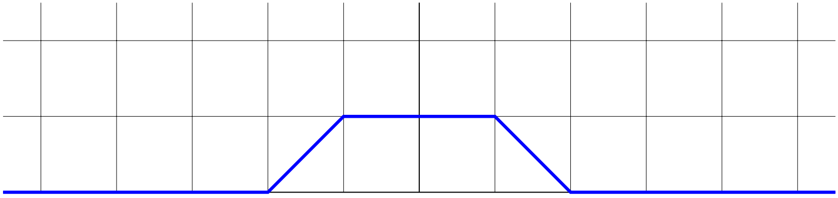
$$(e_j)_{j \in J} \in \mathrm{ApproxId}(C_0(\mathbb{R}))$$

$$\Updownarrow$$

$$\forall K \in \mathcal{K} \qquad e_j|_K \rightrightarrows 1_K$$

Example of approximate identity in $C_0(\mathbb{R})$

$$e_j(t) := \begin{cases} 1, & |t| \le j; \\ j+1-|t|, & j < |t| \le j+1; \\ 0, & |t| > j+1. \end{cases}$$



Graph of e_1

Example of approximate identity in $C_0(\mathbb{R})$

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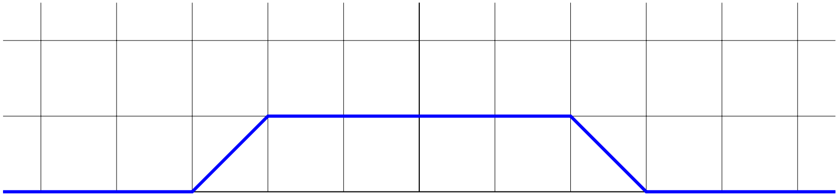
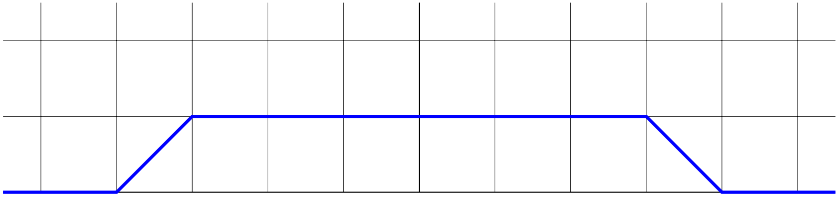


Graph of e_2

Example of approximate identity in $C_0(\mathbb{R})$

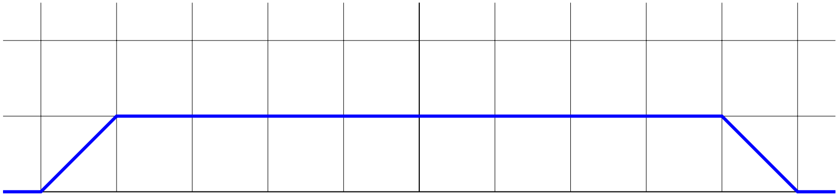
$$e_j(t) := \begin{cases} 1, & |t| \leq j; \\ j + 1 - |t|, & j < |t| \leq j + 1; \\ 0, & |t| > j + 1. \end{cases}$$



Graph of e_3

Example of approximate identity in $C_0(\mathbb{R})$

$$e_j(t) := \begin{cases} 1, & |t| \le j; \\ j+1-|t|, & j < |t| \le j+1; \\ 0, & |t| > j+1. \end{cases}$$



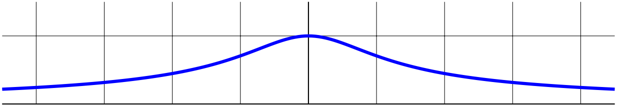
Graph of e_4

Criterion of approximate invertibility in $\mathcal{A} = C_0(\mathbb{R})$

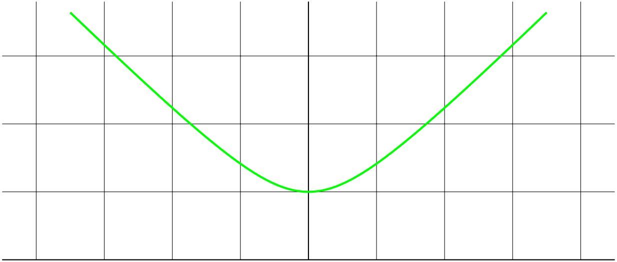
$$\begin{tikzcd}
 & f \in \mathrm{ApproxInv}(\mathcal{A}) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\mathrm{clos}(f\,\mathcal{A}) = \mathcal{A} \arrow[rr, leftrightarrow] & & \forall t \in \mathbb{R} \quad f(t) \neq 0
\end{tikzcd}$$

Example of approximately invertible element in $C_0(\mathbb{R})$

$$f(t) = \frac{1}{\sqrt{1+t^2}}$$



$1/f$ is unbounded



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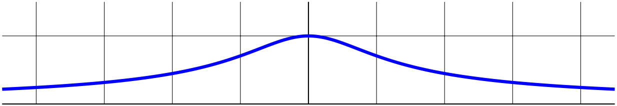
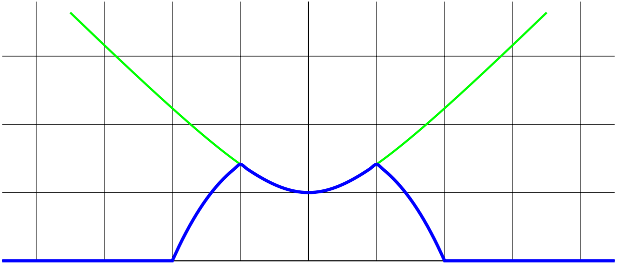
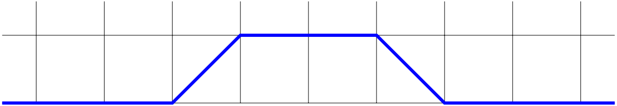


$$g_1 := e_1/f$$



$$f\,g_1 = e_1$$



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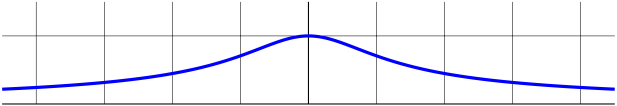
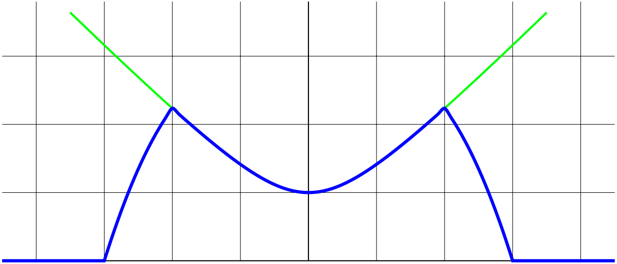
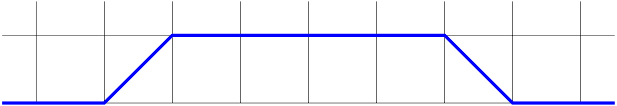


$$g_2 := e_2/f$$



$$f\,g_2 = e_2$$



Example of approximately invertible element in $C_0(\mathbb{R})$

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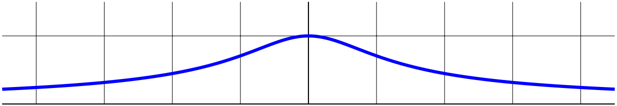
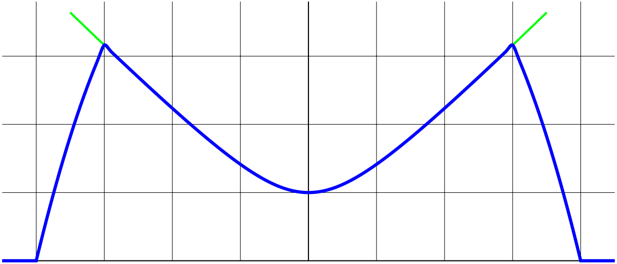
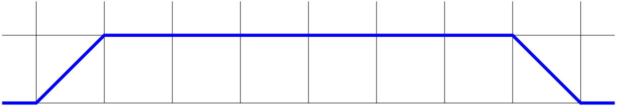


$$g_3 := e_3/f$$



$$f\,g_3 = e_3$$



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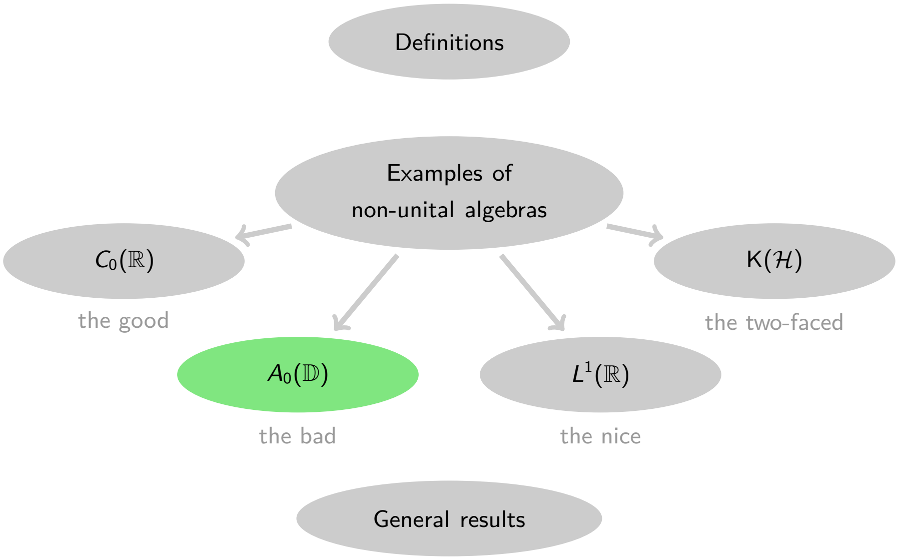

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Small disk algebra

$$\mathbb{D} := \{z \in \mathbb{C}:\ \ |z| < 1\}.$$

Disk algebra $A :=$ all continuous functions $\mathrm{clos}(\mathbb{D}) \to \mathbb{C}$
that are holomorphic in $\mathbb{D}$:

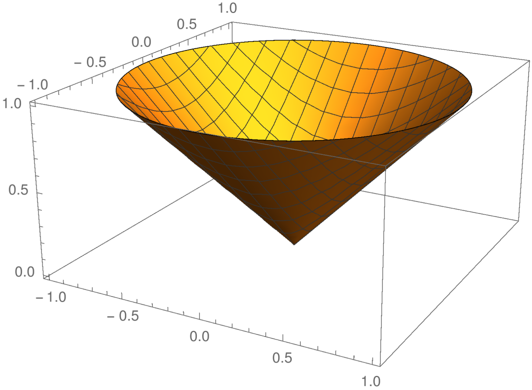
$$A := \big\{f \in C(\mathrm{clos}(\mathbb{D})):\quad f|_{\mathbb{D}} \in H(\mathbb{D})\big\}.$$

Small disk algebra $A_0 :=$ all continuous functions $\mathrm{clos}(\mathbb{D}) \to \mathbb{C}$
that are holomorphic in $\mathbb{D}$ and vanish at 0:

$$A_0 := \big\{f \in A:\quad f(0) = 0\big\}.$$

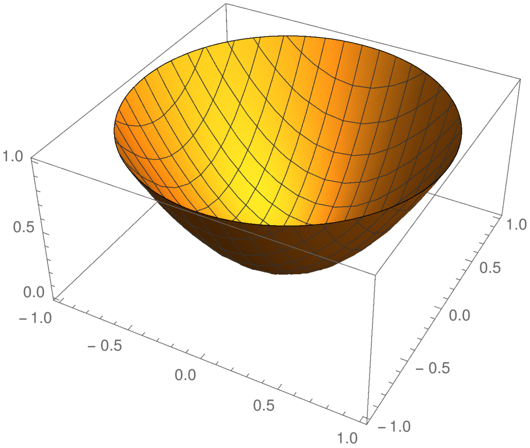
A_0 is a non-unital closed subalgebra of $C(\mathrm{clos}(\mathbb{D}))$.
A_0 is generated by the monomial $g(z) := z$.

Generator of the small disk algebra



Plot of the absolute value of $g(z) := z$.

Example of element of A_0



Plot of the absolute value of $f(z) := z^2$.

Collapse of ideals and loss of identity

Schwarz lemma easily yields the following properties of A_0.

Proposition

For every f in A_0, the ideal $\overline{f A_0}$ is not dense in A_0.

Proposition

The algebra A_0 has no approximate identities.

As a consequence, A_0 has no approximately invertible elements.

There are elements in A_0 with non-vanishing Gelfand transforms

For every point t in $\mathrm{clos}(\mathbb{D})$, the evaluation at t is a character of A:

$$\varphi_t(f) = f(t) \qquad (t \in \mathrm{clos}(\mathbb{D}), \quad f \in A).$$

Moreover, all characters of the disk algebra all evaluation functionals:

$$\mathcal{M}_A \equiv \mathrm{clos}(\mathbb{D}).$$

Therefore,

$$\mathcal{M}_{A_0} \equiv \mathrm{clos}(\mathbb{D}) \setminus \{0\}.$$

Recall that $g(z) \coloneqq z$.
The Gelfand transform of g does not vanish, but $g \notin \mathrm{ApproxInv}(A_0)$.

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Convolution algebra $L^1(\mathbb{R})$

Provide the Banach space $L^1(\mathbb{R})$ with the convolution operation:

$$(f * g)(x) \coloneqq \int_{\mathbb{R}} f(x-y)g(y)\,dy.$$

$L^1(\mathbb{R})$ is a non-unital commutative Banach algebra.

For every f in $L^1(\mathbb{R})$, denote by $\widehat{f}$ the Fourier transform of f:

$$\widehat{f}(\xi) \coloneqq \int_{\mathbb{R}} f(x)\,e^{-\mathrm{i}\,\xi\,x}\,dx.$$

Convolution theorem:

$$\widehat{f * g} = \widehat{f}\,\widehat{g}.$$

The Fourier transform on $L^1(\mathbb{R})$ coincides with the Gelfand transform.

Dirac sequences

Definition

A sequence $(e_j)_{j\in\mathbb{N}}$ in $L^1(\mathbb{R})$ is a Dirac sequence if:

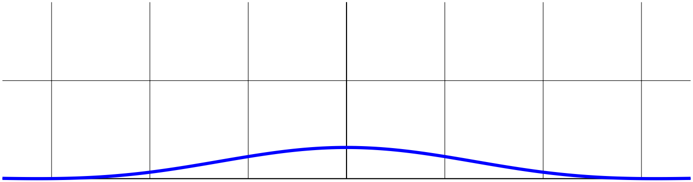
1. $e_j(x) \geq 0$ for every $x \in \mathbb{R}$, $j \in \mathbb{N}$;
2. for every $j \in \mathbb{N}$,
$$\int_{\mathbb{R}} e_j(x)\,dx = 1;$$
3. for every $\delta > 0$,
$$\lim_{j\to\infty} \int_{|x|\geq\delta} e_j(x)\,dx = 0.$$

Every Dirac sequence is an approximate identity in $L^1(\mathbb{R})$.

Example of Dirac sequence (Fejér kernel)

$$e_j(x) = \frac{(\sin(jx))^2}{\pi j x^2}.$$

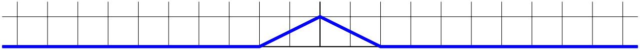
$j = 1$



In this example the supports of $\widehat{e_j}$ are compact:

$$\widehat{e_j}(t) = \begin{cases} 1 - \frac{|t|}{2j}, & |t| \leq 2j; \\ 0, & |t| > 2j. \end{cases}$$

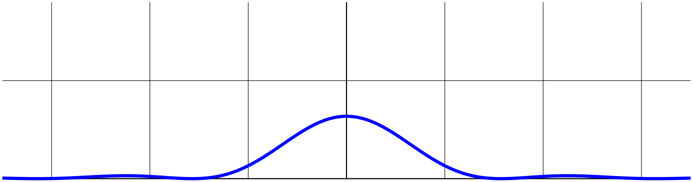
$j = 1$



Example of Dirac sequence (Fejér kernel)

$$e_j(x) = \frac{(\sin(jx))^2}{\pi j x^2}.$$

$j = 2$



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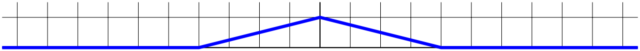

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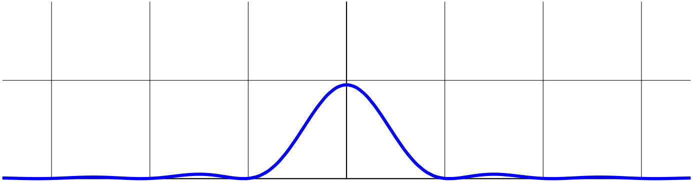
$j = 3$



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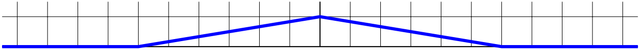

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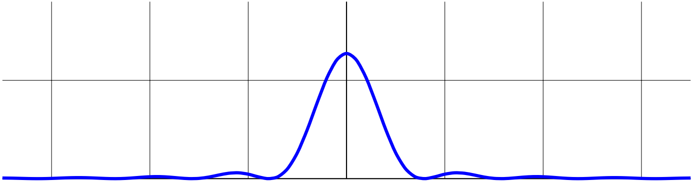
$j = 4$



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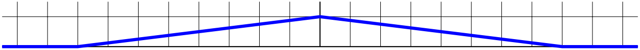
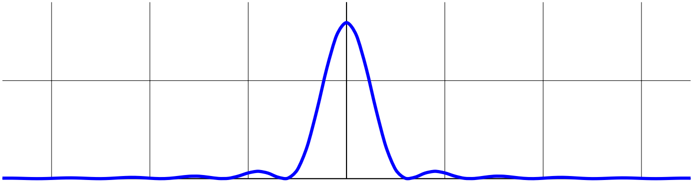


Example of Dirac sequence (Fejér kernel)

$$e_j(x) = \frac{(\sin(jx))^2}{\pi j x^2}.$$

$j = 5$



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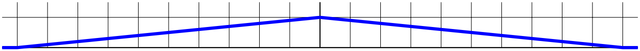


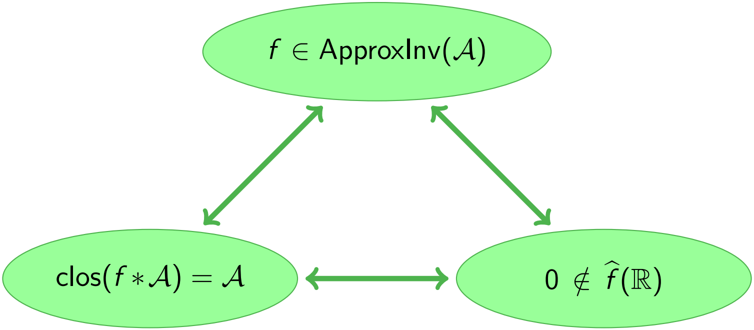
Wiener's Division Lemma

$$f \in L^1(\mathbb{R})$$
$$\operatorname{supp}(\widehat{f}\,) \in \mathcal{K}$$

$$g \in L^1(\mathbb{R})$$
$$\forall x \in \operatorname{supp}(\widehat{f}\,) \quad \widehat{g}(x) \neq 0$$

$$\exists h \in L^1(\mathbb{R})$$
$$f = g * h$$

Criterion of approximate invertibility in $\mathcal{A} = L^1(\mathbb{R})$

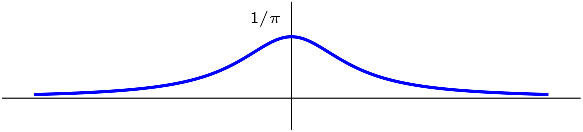


Proof of the implication $\quad 0 \notin \widehat{f}(\mathbb{R}) \implies f \in \mathrm{ApproxInv}(\mathcal{A})$**.**
Let $(e_j)_{j \in \mathbb{N}}$ be a Dirac sequence such that $\mathrm{supp}(\widehat{e_j}) \in \mathcal{K}$.
Using Wiener's Division Lemma we construct $g_j \in L^1(\mathbb{R})$ such that

$$e_j = f * g_j. \qquad \square$$

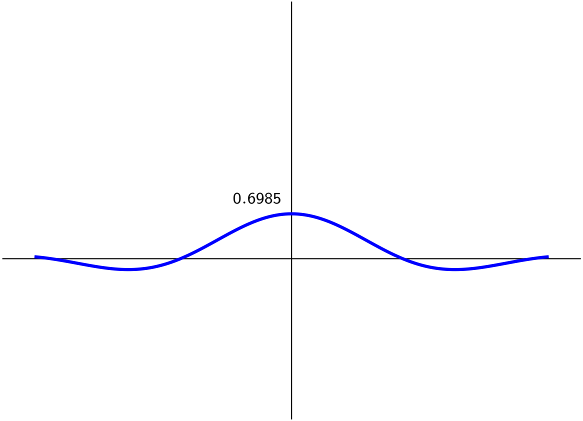
Example of element in ApproxInv($L^1(\mathbb{R})$)

$$f(x) = \frac{1}{\pi(1+x^2)}$$



h_1 such that

$$f * h_1 = e_1$$



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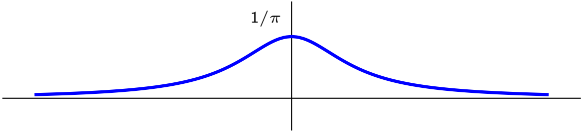
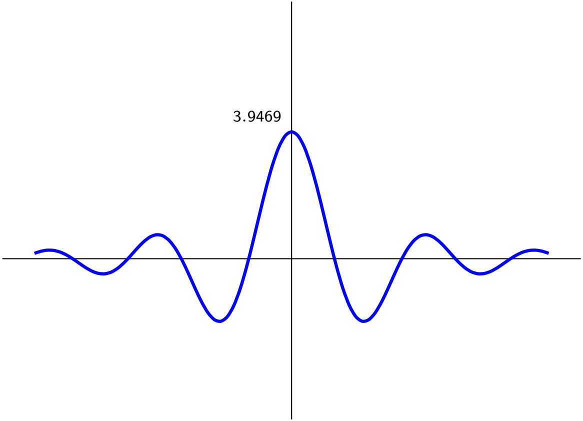


h_2 such that

$$f * h_2 = e_2$$



Example of element in ApproxInv($L^1(\mathbb{R})$)

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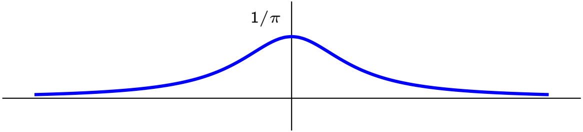
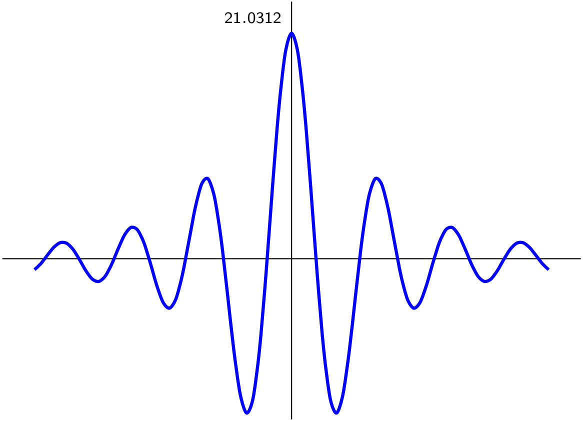
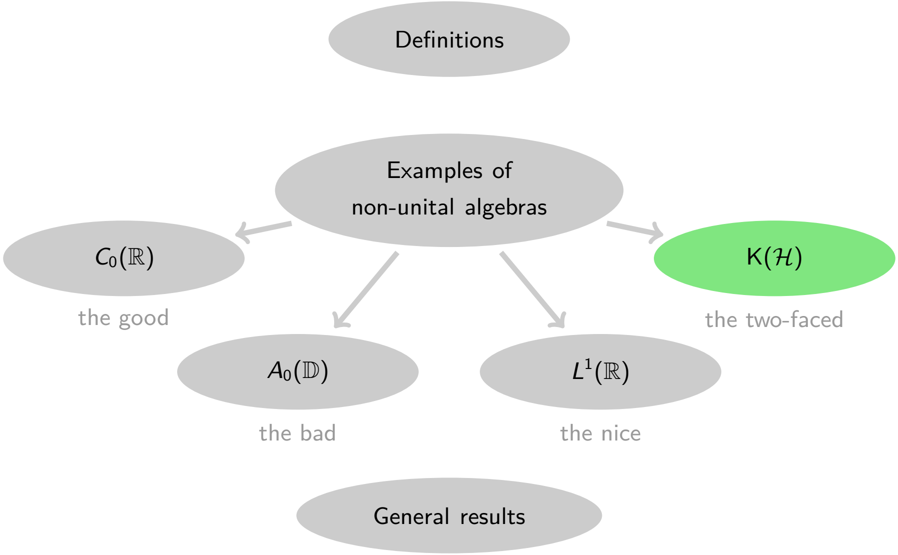


h_3 such that

$$f * h_3 = e_3$$



Outline


Definitions
Examples of
non-unital algebras
$C_0(\mathbb{R})$
the good
$A_0(\mathbb{D})$
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the nice
$K(\mathcal{H})$
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General results

Algebra of compact operators in a Hilbert space

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space.

$\mathrm{K}(\mathcal{H}) :=$ the compact operators acting in $\mathcal{H}$.

$\mathrm{K}(\mathcal{H})$ is a non-unital non-commutative Banach algebra.
Moreover, $\mathrm{K}(\mathcal{H})$ is C*-algebra.

Recall one important property of compact operators.
If $(S_n)_{n=1}^{\infty}$ is a sequence of bounded linear operators in $\mathcal{H}$,
$S_n v \to 0$ for each v in $\mathcal{H}$ and $T \in \mathrm{K}(\mathcal{H})$, then

$$\|S_n T\| \to 0, \qquad \|T S_n\| \to 0.$$

Approximate identity associated to an orthonormal basis

Let $(b_n)_{n=1}^{\infty}$ be an orthonormal basis in $\mathcal{H}$. Then

$$\forall v \in \mathcal{H} \qquad v = \sum_{j=1}^{\infty} \langle v, b_j \rangle b_j.$$

For each m in $\{1, 2, 3, \ldots\}$ define $P_m \colon \mathcal{H} \to \mathcal{H}$ by

$$P_m v := \sum_{j=1}^{m} \langle v, b_j \rangle \, b_j.$$

For example,

$$v = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 + \ldots \qquad \mapsto \qquad P_2 v = \alpha_1 b_1 + \alpha_2 b_2.$$

P_m is the orthogonal projection onto $\operatorname{span}(b_1, \ldots, b_m)$.

Approximate identity associated to an orthonormal basis

$$P_m v := \sum_{j=1}^{m} \langle v, b_j \rangle \, b_j.$$

Proposition

$$(P_m)_{m=1}^{\infty} \in \mathrm{ApproxId}(\mathrm{K}(\mathcal{H})).$$

Proof. The sequence $(P_m)_{m \in \mathbb{N}}$ converges strongly to the identity operator:

$$\forall v \in \mathcal{H} \qquad (P_m - I)v \to 0.$$

If $T \in \mathrm{K}(\mathcal{H})$, then

$$\|P_m T - T\| \to 0 \qquad \text{and} \qquad \|TP_m - T\| \to 0. \qquad \square$$

Singular value decomposition of compact operators

Let $T \in \mathrm{K}(\mathcal{H})$, $r = \operatorname{rank}(T)$.
We put $r = +\infty$, if $T(\mathcal{H})$ is infinite-dimensional.
There exist two orthonormal sequences $(a_j)_{j=1}^r$ y $(b_j)_{j=1}^r$
and a sequence $(s_j)_{j=1}^r$ such that

$$s_1 \ge s_2 \ge s_3 \ge \ldots > 0, \qquad \lim_{j\to\infty} s_j = 0,$$

$$Ta_1 = s_1 b_1, \qquad Ta_2 = s_2 b_2, \qquad Ta_3 = s_3 b_3, \qquad \ldots$$

Therefore, for every v in $\mathcal{H}$,

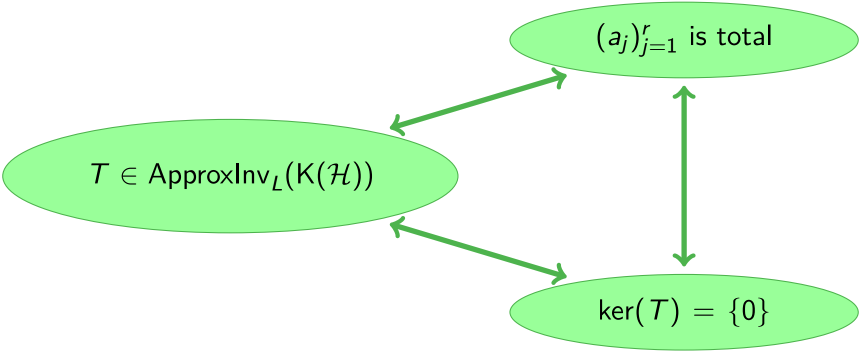
$$Tv = \sum_{j=1}^{r} s_j \langle v, a_j \rangle b_j.$$

Left approximate invertibility in $K(\mathcal{H})$

Let $T \in K(\mathcal{H})$, with the following SVD:

$$Tv = \sum_{j=1}^{r} s_j \langle v, a_j \rangle b_j.$$

Then



Left approximate invertibility in $\mathrm{K}(\mathcal{H})$

Idea of proof

Suppose that $\ker(T) = \{0\}$. Then $r = +\infty$ and $(a_j)_{j=1}^{+\infty}$ is total.
Define U_m:

T	U_3
$a_1 \mapsto s_1 b_1$	$b_1 \mapsto a_1/s_1$
$a_2 \mapsto s_2 b_2$	$b_2 \mapsto a_2/s_2$
$a_3 \mapsto s_3 b_3$	$b_3 \mapsto a_3/s_3$
$a_4 \mapsto s_4 b_4$	$b_4 \mapsto 0$
$a_5 \mapsto s_5 b_5$	$b_5 \mapsto 0$
$\dots$	$\dots$

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Suppose that $\ker(T) = \{0\}$. Then $r = +\infty$ and $(a_j)_{j=1}^{+\infty}$ is total.
Define U_m:

T	U_3	Compute $U_3 T$:
$a_1 \mapsto s_1 b_1$	$b_1 \mapsto a_1/s_1$	
$a_2 \mapsto s_2 b_2$	$b_2 \mapsto a_2/s_2$	
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Left approximate invertibility in $\mathrm{K}(\mathcal{H})$

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Left approximate invertibility in $\mathrm{K}(\mathcal{H})$

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Then $U_m T =$

Left approximate invertibility in $\mathrm{K}(\mathcal{H})$

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$\dots$	$\dots$	

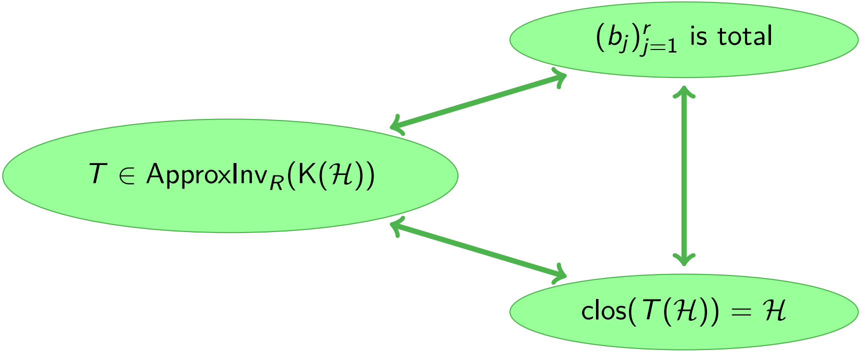
Then $U_m T = P_m$, so $T \in \mathrm{ApproxInv}_L(\mathrm{K}(\mathcal{H}))$.

Right approximative invertibility in $\mathsf{K}(\mathcal{H})$

Let $T \in \mathsf{K}(\mathcal{H})$, and let T have the following SVD:

$$Tv = \sum_{j=1}^{r} s_j \langle v, a_j \rangle b_j.$$

Then



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General results

Theorem 2

Let $\mathcal{A}$ be a commutative Banach algebra and $x \in \mathcal{A}$.
Suppose that $\mathcal{A}$ has an approximate identity.

$$x \in \mathrm{ApproxInv}(\mathcal{A})$$

$$\Updownarrow$$

$$\mathrm{clos}(x\mathcal{A}) = \mathcal{A}$$

$$\Downarrow$$

$$\forall \varphi \in \mathcal{M}_{\mathcal{A}} \quad \widehat{x}(\varphi) \neq 0$$

Idea of proof: if $x\mathcal{A}$ is dense in $\mathcal{A}$, then x is approximately invertible

Let $(e_j)_{j \in J}$ be an approximate identity of $\mathcal{A}$.
Consider the product of directed sets $J \times \mathbb{N}$.
For every $j \in J$ and every k in $\mathbb{N}$ choose $u_{j,k} \in \mathcal{A}$ such that

$$\|xu_{j,k} - e_j\| < \frac{1}{k}.$$

Then $(xu_{j,k})_{(j,k) \in J \times \mathbb{N}} \in \mathrm{ApproxId}(\mathcal{A})$.

Theorem 2 for non-commutative case

Suppose that $\mathcal{A}$ is a Banach algebra and $x \in \mathcal{A}$.
Moreover, let $\mathcal{A}$ have an approximate identity.

$$x \in \mathrm{ApproxInv}_R(\mathcal{A})$$

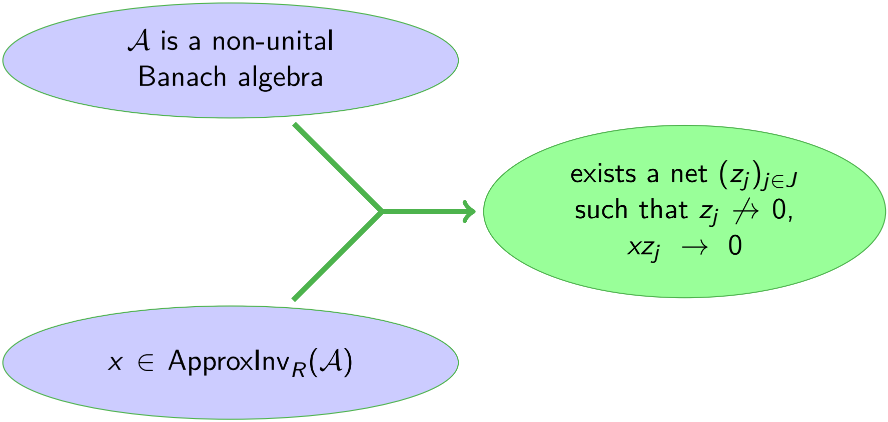
$$\Updownarrow$$

$$\mathrm{clos}(x\mathcal{A}) = \mathcal{A}$$

$$\Downarrow$$

x does not belong to any
maximal modular right ideal

Relation with topological zero divisors (Theorem 3)



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Ďakujem vám za pozornosť!