Determinants, eigenvalues and eigenvectors of tetradiagonal real Toeplitz matrices

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Outline

Tetradiagonal real Toeplitz matrices

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Results for a particular case

We consider real nonsymmetric Toeplitz matrices $T_n(a)$ generated by Laurent polynomials of the form

$$a(t) = a_{-1}t^{-1} + a_0 + a_1 t + a_2 t^2,$$

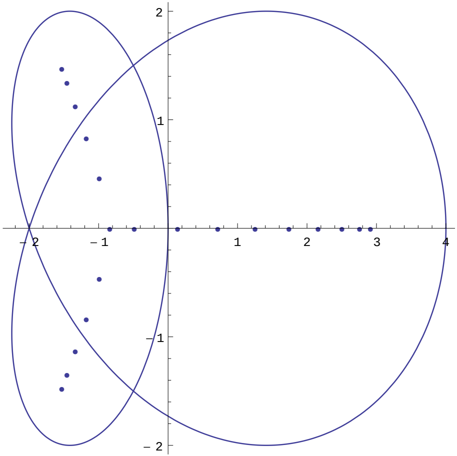
where $a_{-1}, a_0, a_1, a_2 \in \mathbb{R}$, $a_{-1} \neq 0$, $a_2 \neq 0$.

For example,

$$T_6(a) = \begin{bmatrix} a_0 & a_{-1} & 0 & 0 & 0 & 0 \\ a_1 & a_0 & a_{-1} & 0 & 0 & 0 \\ a_2 & a_1 & a_0 & a_{-1} & 0 & 0 \\ 0 & a_2 & a_1 & a_0 & a_{-1} & 0 \\ 0 & 0 & a_2 & a_1 & a_0 & a_{-1} \\ 0 & 0 & 0 & a_2 & a_1 & a_0 \end{bmatrix}.$$

Example: $a(t) = t^{-1} + t + 2t^2$

The generating symbol a is considered as a function $\mathbb{T} \to \mathbb{C}$.
Its image is an analytic curve in the complex plane.



Schmidt and Spitzer (1960) proved that as $n \to \infty$,
the spectra of $T_n(a)$ fill a certain "skeleton" of this curve.

Determinants and eigenvectors in terms of $S_n(z_1, z_2, z_3)$

For every $\lambda \in \mathbb{C}$ we denote by $z_1(\lambda), z_2(\lambda), z_3(\lambda)$ the roots of $a(z) - \lambda$:

$$a(z) - \lambda = z^{-1}(z - z_1(\lambda))(z - z_2(\lambda))(z - z_3(\lambda)).$$

Then

$$\det T_n(a - \lambda) = S_n(z_1(\lambda), z_2(\lambda), z_3(\lambda)),$$

where S_n is the following symmetric polynomial of three variables:

$$S_n(z_1, z_2, z_3) = \sum_{\substack{p_1, p_2, p_3 \geq 0 \\ p_1 + p_2 + p_3 = n}} z_1^{n-p_1} z_2^{n-p_2} z_3^{n-p_3}.$$

If λ is an eigenvalue of $T_n(a)$ and

$$v_m = (z_1(\lambda) z_2(\lambda) z_3(\lambda))^{n-m-1} S_m(z_1(\lambda), z_2(\lambda), z_3(\lambda)),$$

then $v = [v_m]_{m=1}^n$ is an eigenvector associated to λ.

$S_n(z_1, z_2, z_3)$ through $A_n(z_1, z_2, z_3)$

The symmetric polynomial

$$S_n(z_1, z_2, z_3) = \sum_{\substack{p_1, p_2, p_3 \geq 0 \\ p_1 + p_2 + p_3 = n}} z_1^{n-p_1} z_2^{n-p_2} z_3^{n-p_3}$$

may be written as a quotient of two antisymmetric polynomials:

$$\frac{A_n(z_1, z_2, z_3)}{V_n(z_1, z_2, z_3)},$$

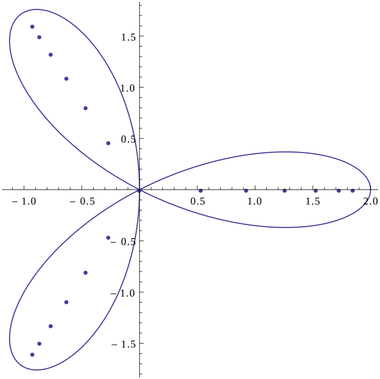
where $V_n(z_1, z_2, z_3)$ is the polynomial of Vandermonde

$$V_n(z_1, z_2, z_3) = \prod_{1 \leq j < k \leq 3} (z_k - z_j) = (z_2 - z_1)(z_3 - z_1)(z_3 - z_2),$$

and $A_n(z_1, z_2, z_3)$ is the antisymmetrization of $z_1^{n+1} z_2^{n+2}$:

$$\begin{aligned} A_n(z_1, z_2, z_3) = z_1^{n+1} z_2^{n+2} + z_2^{n+1} z_3^{n+2} + z_3^{n+1} z_1^{n+2} \\ - z_1^{n+2} z_2^{n+1} - z_2^{n+2} z_3^{n+1} - z_3^{n+2} z_1^{n+1}. \end{aligned}$$

Results for a particular case: $a(t) = t^{-1} + t^2$



In this example the spectra of $T_n(a)$ are symmetric under $\frac{2\pi}{3}$-rotation, the Schmidt and Spitzer set is $\{\rho\, e^{\frac{2k\pi i}{3}} : \ \rho \in [0, 3/\sqrt[3]{4}], \ k \in \{0, 1, 2\}\}$, and it is sufficient to consider only one branch: $\lambda \in [0, 3/\sqrt[3]{4}]$.

Results for a particular case: $a(t) = t^{-1} + t^2$

If $a(t) = t^{-1} + t^2$ and $\lambda \in [0, 3/\sqrt[3]{4}]$, then it is convenient to express $\det(T_n(a - \lambda))$ in terms of the parameter $\alpha \in [0, \pi/3]$ related to λ by

$$\lambda = \frac{1}{\sqrt[3]{4}}(3 - \tan^2 \alpha)\,(\cos \alpha)^{4/3}\,.$$

The expression $S_n(z_1(\lambda), z_2(\lambda), z_3(\lambda))$ may be written as

$$C_n(\alpha)\Big(\sin((n+1)\alpha + \theta(\alpha)) + R_n(\alpha)\Big),$$

where $C_n(\alpha) \neq 0$, $\theta(\alpha) = \arctan \dfrac{\tan \alpha}{3}$, and

$$R_n(\alpha) = \frac{(-1)^n \tan(\alpha)}{\sqrt{9 + (\tan \alpha)^2}} \cdot \left(\frac{1}{2 \cos \alpha}\right)^{n+1}.$$

When $\lambda \geq \delta > 0$, the term $R_n(\alpha)$ decays exponentially.