Approximately invertible elements in non-unital normed algebras

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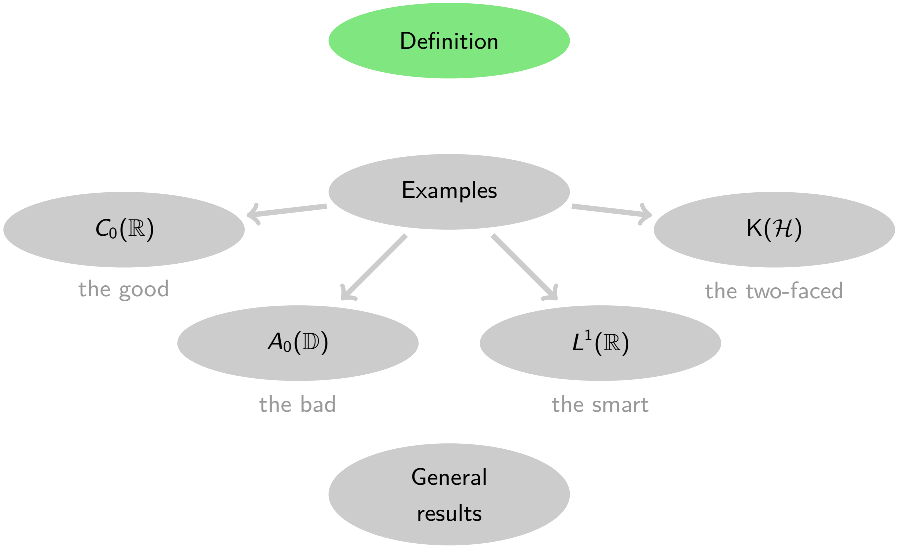
Outline



Definition
Examples
$C_0(\mathbb{R})$
the good
$\mathsf{K}(\mathcal{H})$
the two-faced
$A_0(\mathbb{D})$
the bad
$L^1(\mathbb{R})$
the smart
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Definition of normed algebra (review)

Let $\mathcal{A}$ be a normed complex vector space and at the same time an algebra. It is said that $\mathcal{A}$ is a normed algebra if the norm in $\mathcal{A}$ is *submultiplicative*:

$$\forall a, b \in \mathcal{A} \qquad \|ab\| \leq \|a\| \, \|b\|.$$

A normed algebra $\mathcal{A}$ is called a Banach algebra
if $\mathcal{A}$ is *complete* with respect to the distance induced by the norm.

Main examples of Banach algebras:

- Algebra $C_b(T, \mathbb{C})$ of bounded continuous functions $T \to \mathbb{C}$, where T is a topological space.
- Algebra $B(X, X)$ of bounded linear operators acting in a Banach space X.

Definition of net (review)

The concept of *nets* generalizes the concept of *sequences*.

Let J be a set and $\succeq$ be a partial order on J.
It is said that $(J, \succeq)$ is a directed set if

$$\forall p, q \in J \qquad \exists r \in J \qquad (r \succeq p) \quad \wedge \quad (r \succeq q).$$

A net in a topological space (X, τ) is a function $s \colon J \to X$, there $(J, \preceq)$ is a directed set. We shall write $(s_j)_{j \in J}$ instead of s.

Let (X, τ) be a topological space, $(s_j)_{j \in J}$ be a net and $y \in X$.
It is said that the net $(s_j)_{j \in J}$ converges to y iff

$$\forall V \in \tau_y \qquad \exists k \in J \qquad \forall j \succeq k \qquad s_j \in V.$$

Definition

Let $\mathcal{A}$ be a Banach algebra.

Definition (approximate identity)

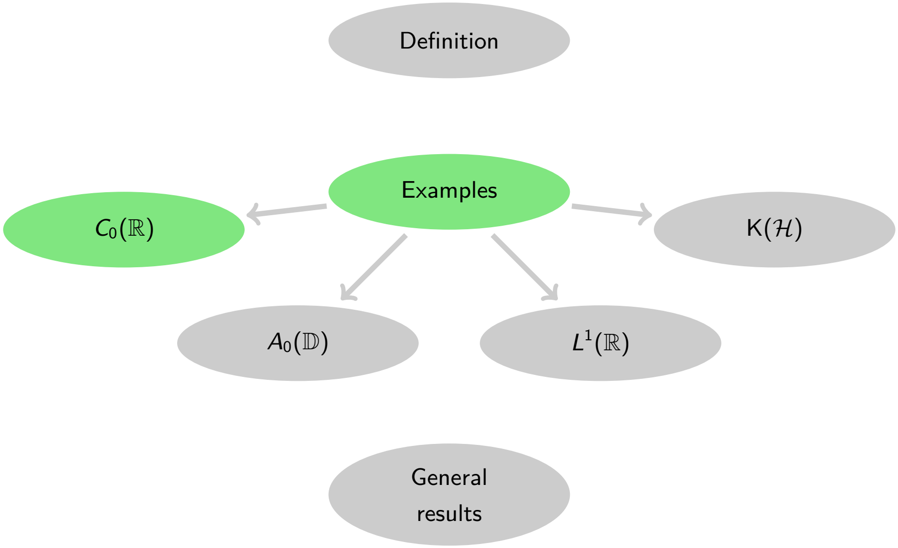
A net $(e_j)_{j\in J}$ in $\mathcal{A}$ is called an approximate identity in $\mathcal{A}$ if for every $a \in \mathcal{A}$

$$\lim_{j\in J} a e_j = a, \qquad \lim_{j\in J} e_j a = a.$$

Definition (right approximately invertible elements)

Let $x \in \mathcal{A}$. We say that x is right approximately invertible if there exists a net $(u_j)_{j\in J}$ such that the net $(x u_j)_{j\in J}$ is an approximate identity in $\mathcal{A}$.

Outline



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Approximate identities in $C_0(\mathbb{R})$

Denote by $\mathcal{K}$ the set of all compact subsets of $\mathbb{R}$.

For every $K \in \mathcal{K}$ denote by 1_K the constant function $K \to \mathbb{C}$ defined by

$$1_K(t) = 1 \qquad (t \in K).$$

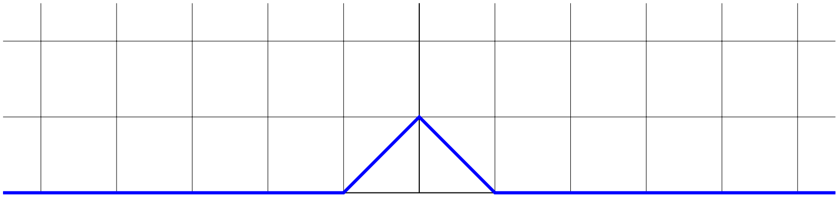
Proposition

Let $(e_j)_{j \in J}$ be a net in $C_0(\mathbb{R})$. Then

$$(e_j)_{j \in J} \text{ is an appr.id.} \qquad \Longleftrightarrow \qquad \forall K \in \mathcal{K} \qquad e_j|_K \rightrightarrows 1_K.$$

Example of approximate identity in $C_0(\mathbb{R})$

$$e_j(t) := \begin{cases} 1, & |t| \leq j; \\ j+1-|t|, & j < |t| \leq j+1; \\ 0, & |t| > j+1. \end{cases}$$



Plot of e_0

Example of approximate identity in $C_0(\mathbb{R})$

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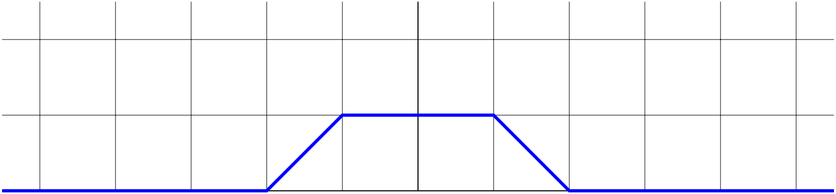
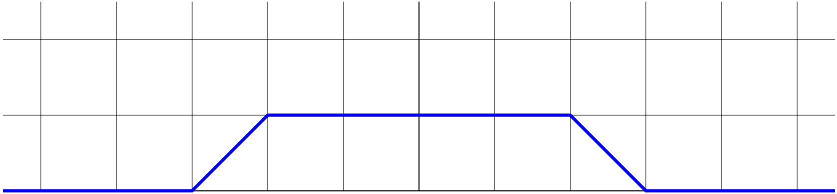


Plot of e_1

Example of approximate identity in $C_0(\mathbb{R})$

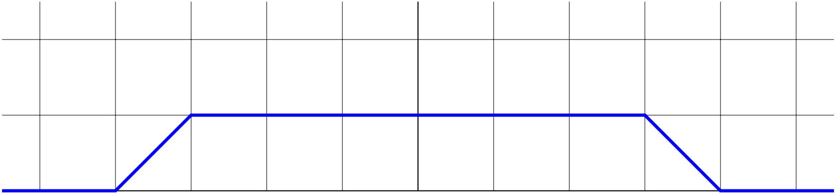
$$e_j(t) := \begin{cases} 1, & |t| \le j; \\ j + 1 - |t|, & j < |t| \le j + 1; \\ 0, & |t| > j + 1. \end{cases}$$



Plot of e_2

Example of approximate identity in $C_0(\mathbb{R})$

$$e_j(t) := \begin{cases} 1, & |t| \le j; \\ j + 1 - |t|, & j < |t| \le j + 1; \\ 0, & |t| > j + 1. \end{cases}$$



Plot of e_3

Approximately invertible elements in $C_0(\mathbb{R})$

Proposition

Let $f \in C_0(\mathbb{R})$*.*
Then the following conditions are equivalent:

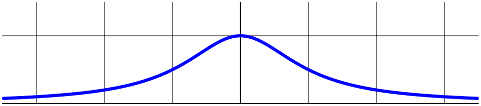
(a) f *is approximately invertible in* $C_0(\mathbb{R})$*.*

(b) $f\mathbb{R}$ *is dense in* $C_0(\mathbb{R})$*.*

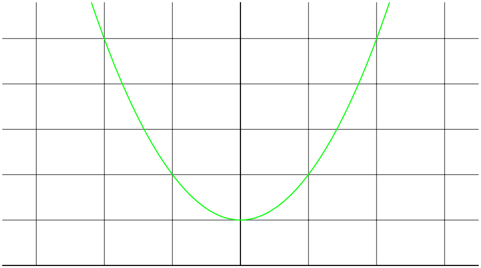
(c) $f(t) \neq 0$ *for every* $t \in \mathbb{R}$*.*

Example of approximately invertible element in $C_0(\mathbb{R})$

$$f(t) = \frac{1}{1+t^2}$$



$1/f$ is not bounded



Example of approximately invertible element in $C_0(\mathbb{R})$

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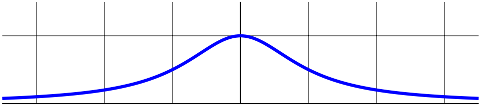
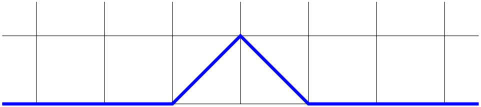
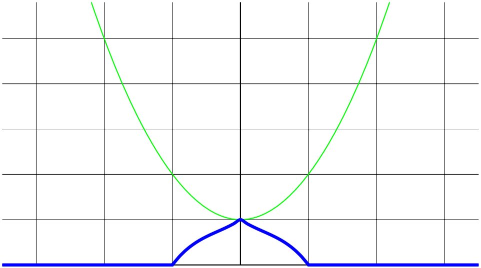


e_0

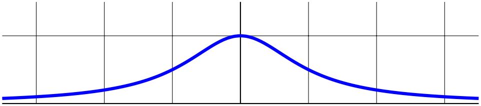


$g_0 := e_0/f$

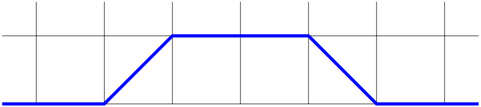


Example of approximately invertible element in $C_0(\mathbb{R})$

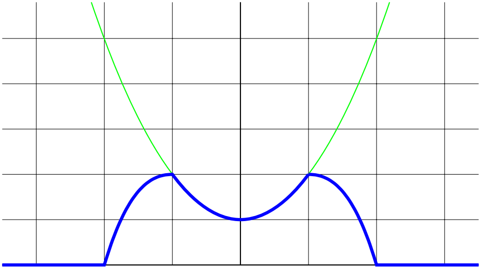
$$f(t) = \frac{1}{1+t^2}$$



e_1

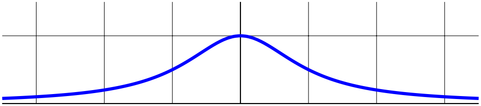


$g_1 := e_1/f$

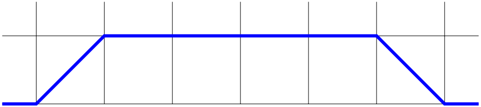


Example of approximately invertible element in $C_0(\mathbb{R})$

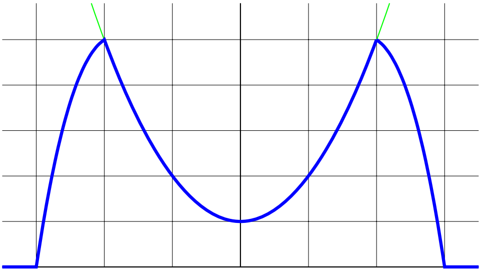
$$f(t) = \frac{1}{1+t^2}$$



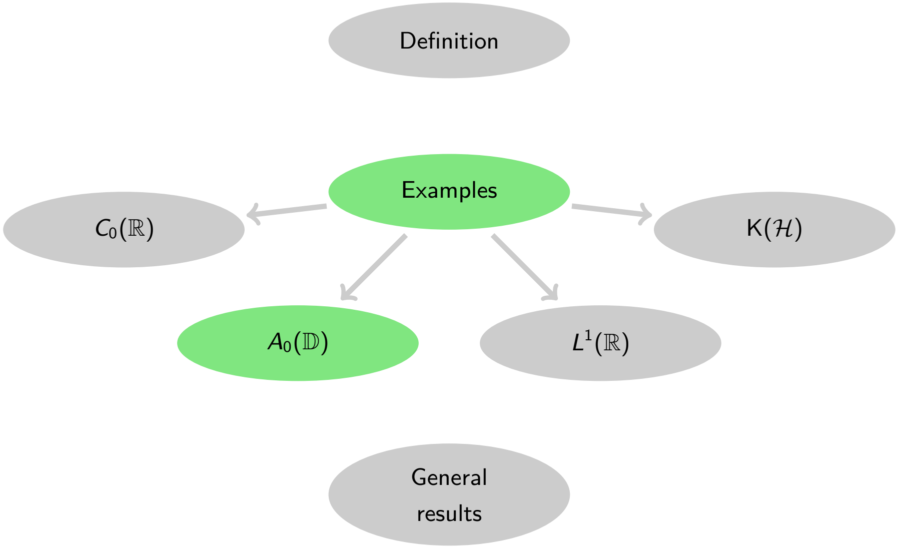
e_2



$g_2 := e_2/f$



Outline



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Small disk algebra

Denote by A_0 the algebra of all continuous functions $\overline{\mathbb{D}} \to \mathbb{C}$ that are analytic in $\mathbb{D}$ and vanish at 0:

$$A_0 := \{f \in C(\overline{\mathbb{D}}) \colon \quad f|_{\mathbb{D}} \in H(\mathbb{D}) \quad \wedge \quad f(0) = 0\}.$$

In other words, A_0 is the non-unital closed subalgebra of $C(\overline{\mathbb{D}})$ generated by the monomial

$$g(z) := z.$$

Lemma

If $f \in A_0$, then for every $z \in \mathbb{D}$

$$|f(z)| \le |z| \, \|f\|_\infty.$$

Proof. Follows from Schwarz Lemma. □

Small disk algebra has no approximate identities

Lemma

For every $f \in A_0$,

$$\sup_{1/2 \leq |z| \leq 1} |f(z) - 1| \geq \frac{1}{3}.$$

Idea of proof. If $\|f\|_\infty \leq \frac{4}{3}$, then $|f(1/2)| \leq \frac{2}{3}$. □

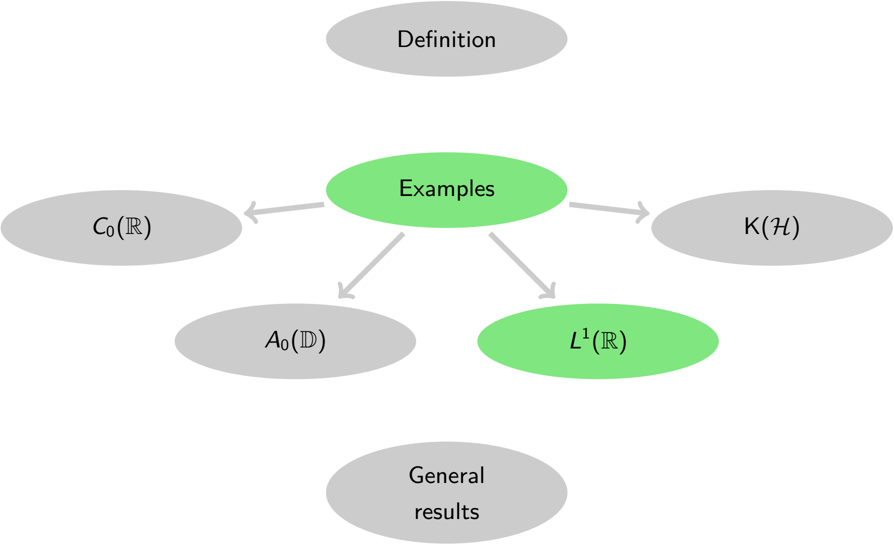
Proposition

Define $g \colon \overline{\mathbb{D}} \to \mathbb{C}$ *by* $g(z) = z$. *Then the ideal* gA_0 *is not dense in* A_0.

Proposition

The algebra A_0 *has no approximate identities.*

Outline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Dirac sequences

In this section we consider $L^1(\mathbb{R})$ with the convolution operation $*$.
It is a non-unital conmutative algebra.
For every $f \in L^1(\mathbb{R})$ denote by $\widehat{f}$ the Fourier transform of f.

Definition

A sequence $(e_j)_{j\in\mathbb{N}}$ in $L^1(\mathbb{R})$ is a *Dirac sequence* if:

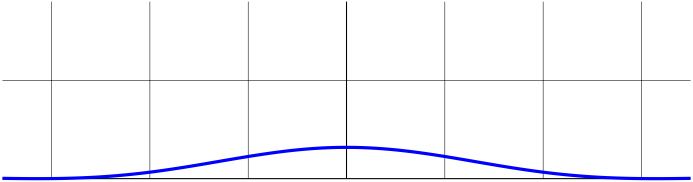
1. $e_j(x) \geq 0$ for every $x \in \mathbb{R}$, $j \in \mathbb{N}$.
2. $\int_{\mathbb{R}} e_j(x)\, dx = 1$ for every $j \in \mathbb{N}$.
3. For every $\delta > 0$,
$$\lim_{j\to\infty} \int_{|x|\geq\delta} e_j(x)\, dx = 0.$$

It is known that every Dirac sequence is an approximate identity in $L^1(\mathbb{R})$.

Example of Dirac sequence

$$e_j(x) = \frac{(\sin(jx))^2}{\pi j x^2}.$$

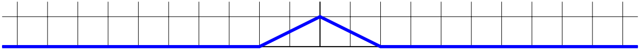
$j = 1$



In this example the support of $\widehat{e}_j$ is compact for every $j \in \mathbb{N}$:

$$\widehat{e}_j(t) = \begin{cases} 1 - \frac{|t|}{2j}, & |t| \leq 2j; \\ 0, & \text{otherwise.} \end{cases}$$

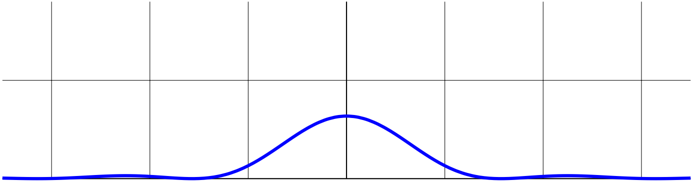
$j = 1$



Example of Dirac sequence

$$e_j(x) = \frac{(\sin(jx))^2}{\pi j x^2}.$$

$j = 2$



In this example the support of $\widehat{e_j}$ is compact for every $j \in \mathbb{N}$:

$$\widehat{e_j}(t) = \begin{cases} 1 - \frac{|t|}{2j}, & |t| \le 2j; \\ 0, & \text{otherwise.} \end{cases}$$

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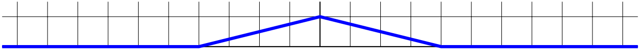
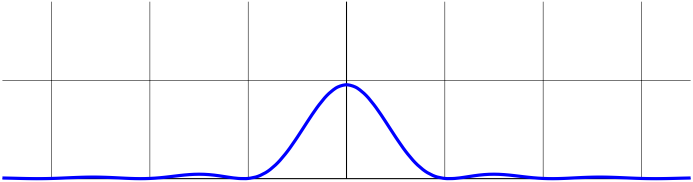


Example of Dirac sequence

$$e_j(x) = \frac{(\sin(jx))^2}{\pi j x^2}.$$

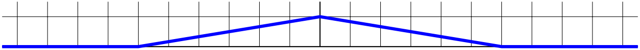
$j = 3$



In this example the support of $\widehat{e_j}$ is compact for every $j \in \mathbb{N}$:

$$\widehat{e_j}(t) = \begin{cases} 1 - \frac{|t|}{2j}, & |t| \leq 2j; \\ 0, & \text{otherwise.} \end{cases}$$

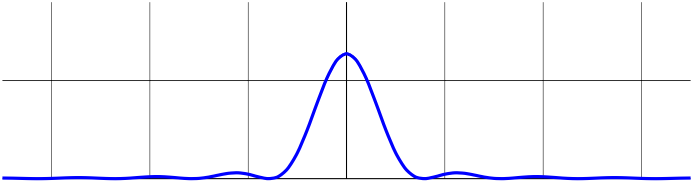
$j = 3$



Example of Dirac sequence

$$e_j(x) = \frac{(\sin(jx))^2}{\pi j x^2}.$$

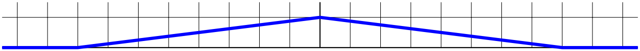
$j = 4$



In this example the support of $\widehat{e}_j$ is compact for every $j \in \mathbb{N}$:

$$\widehat{e}_j(t) = \begin{cases} 1 - \frac{|t|}{2j}, & |t| \leq 2j; \\ 0, & \text{otherwise.} \end{cases}$$

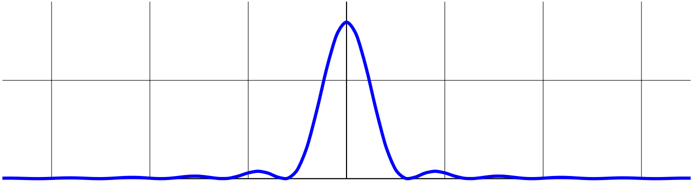
$j = 4$



Example of Dirac sequence

$$e_j(x) = \frac{(\sin(jx))^2}{\pi j x^2}.$$

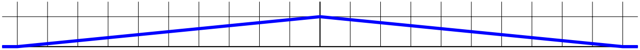
$j = 5$



In this example the support of $\widehat{e_j}$ is compact for every $j \in \mathbb{N}$:

$$\widehat{e_j}(t) = \begin{cases} 1 - \frac{|t|}{2j}, & |t| \le 2j; \\ 0, & \text{otherwise.} \end{cases}$$

$j = 5$



Wiener's Division Lemma

Theorem

Let $f, g \in L^1(\mathbb{R})$ *such that*

- $\text{supp}(\widehat{f})$ *is compact,*
- $\widehat{g}(x) \neq 0$ *for every* $x \in \text{supp}(\widehat{f})$.

Then there exists $h \in L^1(\mathbb{R})$ *such that*

$$f = g * h.$$

Approximately invertible elements in $L^1(\mathbb{R})$

Theorem

Let $f \in L^1(\mathbb{R})$. Then the following conditions are equivalent.

(a) *f is approximately invertible.*

(b) *$f * L^1(\mathbb{R})$ is dense in $L^1(\mathbb{R})$.*

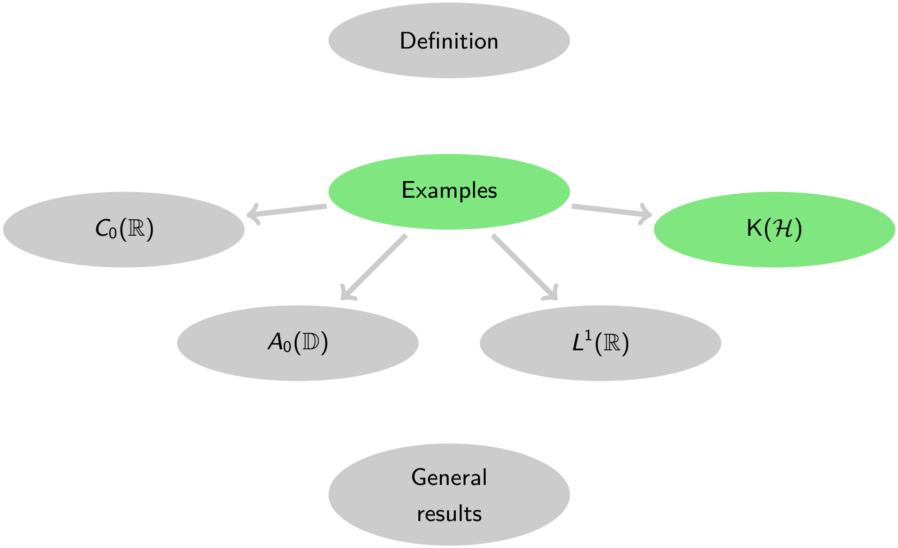
(c) *$\widehat{f}(t) \neq 0$ for every $t \in \mathbb{R}$.*

Proof. We shall prove that (c) implies (a).
Suppose that $\widehat{f}(t) \neq 0$ for every $t \in \mathbb{R}$.
Let $(e_j)_{j \in \mathbb{N}}$ be a Dirac sequence with $\mathrm{supp}(\widehat{e_j}) \in \mathcal{K}$.
For every $j \in \mathbb{N}$, by Wiener's Division Lemma, $\exists g_j \in L^1(\mathbb{R})$ such that

$$e_j = f * g_j.$$

Therefore f is approximately invertible. □

Outline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Algebra of compact operators in a Hilbert space

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space.
Consider the algebra $\mathrm{K}(\mathcal{H})$ of compact operators acting in $\mathcal{H}$.

Proposition

Let $(b_n)_{n\in\mathbb{N}}$ be an orthonormal base of $\mathcal{H}$. For every m in $\mathbb{N}$ let P_m be the orthonormal projection onto the subspace generated by $b_1, \ldots, b_m$:

$$P_m v := \sum_{j=1}^{m} \langle v, b_j \rangle \, b_j.$$

Then $(P_m)_{m\in\mathbb{N}}$ is an approximate identity in $\mathrm{K}(\mathcal{H})$.

Proof.

The sequence $(P_m)_{m\in\mathbb{N}}$ strongly converges to the identity operator I.
Therefore for every $T \in \mathrm{K}(\mathcal{H})$, $\|P_m T - T\| \to 0$ and $\|TP_m - T\| \to 0$. □

Approximate invertibility in the algebra of compact operators

Proposition

Let $T \in \mathrm{K}(\mathcal{H})$*.*

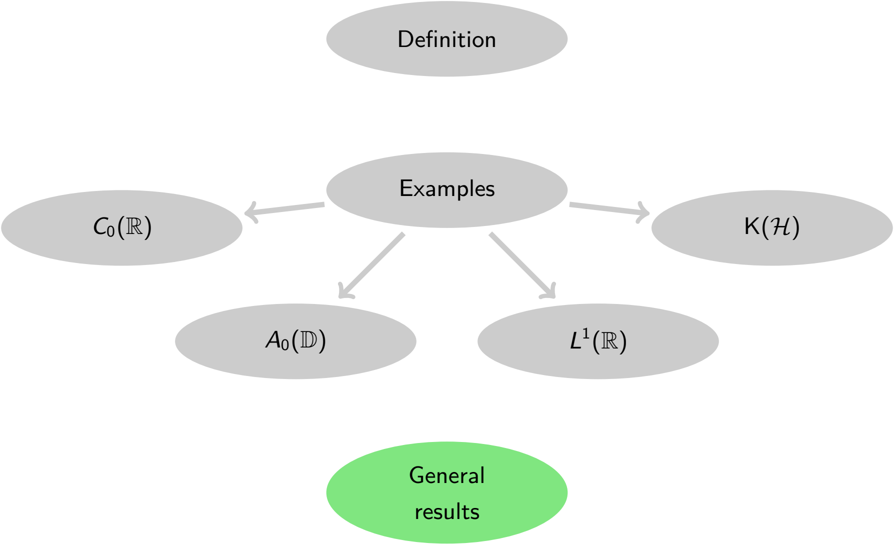
- T *is approximately left invertible* $\iff$ $\ker(T) = \{0\}$*.*
- T *is approximately right invertible* $\iff$ $T(\mathcal{H})$ *is dense in* $\mathcal{H}$*.*

The $\Longleftarrow$ implications may be proved using the previous Proposition and the singular value decomposition of T:

$$Tv = \sum_{j\in\mathbb{N}} s_j \langle v, a_j\rangle b_j,$$

where $s_1 \geq s_2 \geq \ldots,\quad s_j \to 0$ as $j \to \infty$,
$(a_j)_{j\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ are orthonormal sequences in $\mathcal{H}$.

Outline



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Two notes about approximate identities

Proposition

Let $\mathcal{A}$ be a unital normed algebra and e be a unity of $\mathcal{A}$. Then for every directed set J the constant net $(e)_{j\in J}$ is an approximate identity of $\mathcal{A}$.

Proposition

Let $(e_j)_{j\in J}$ be an approximate identity in $\mathcal{A}$ having a limit $f \in \mathcal{A}$. Then f is the unity of $\mathcal{A}$.

Proof. For every $a \in \mathcal{A}$,

$$a = \lim_{j\in J} ae_j = af, \qquad a = \lim_{j\in J} e_j a = fa.$$

□

If x is right approximately invertible, then $x\mathcal{A}$ is dense in $\mathcal{A}$

Proposition

Let x be a right approximately invertible element of $\mathcal{A}$. Then the right ideal $x\mathcal{A}$ is dense in $\mathcal{A}$.

Proof. Let $(u_j)_{j\in J}$ be a net in $\mathcal{A}$ such that $(xu_j)_{j\in J}$ is an appr.id. in $\mathcal{A}$. Given $a \in \mathcal{A}$, consider the net $(xu_j a)_{j\in J}$.

$$x\mathcal{A} \ni x(u_j a) = (xu_j)a \to a. \qquad \square$$

Approximate invertibility and maximal modular ideals

Definition

Let $\mathcal{A}$ be an algebra. A right ideal J is said to be modular if there exists an element v of $\mathcal{A}$ such that

$$\forall x \in \mathcal{A} \qquad vx - x \in J.$$

It is well known that every modular right ideal is contained in a maximal modular right ideal.

Proposition

Let $\mathcal{A}$ be a non-unital normed algebra, $x \in \mathcal{A}$ and $\mathcal{A}x$ is dense in $\mathcal{A}$. Then there is no maximal modular right ideal J such that $x \in J$.

If x is right approximately invertible, then $x \notin \mathcal{A}x$

Proposition

Let $\mathcal{A}$ be a non-unital normed algebra and x be a right approximately invertible element of $\mathcal{A}$. Then $x \notin \mathcal{A}x$.

Proof. Suppose that $x = yx$ for some $y \in \mathcal{A}$. Let $(u_j)_{j\in J}$ be a net in $\mathcal{A}$ such that $(xu_j)_{j\in J}$ is an approximate identity in $\mathcal{A}$. Then

$$\lim_{j\in J} xu_j = \lim_{j\in J} yxu_j = y.$$

Therefore y is a unity of $\mathcal{A}$. Contradiction. □

If $x\mathcal{A}$ is dense in $\mathcal{A}$, then what about x?

Theorem

Let $\mathcal{A}$ be a normed algebra having an approximate identity and $x \in \mathcal{A}$ such that $x\mathcal{A}$ is dense in $\mathcal{A}$.
Then x is approximately right invertible.

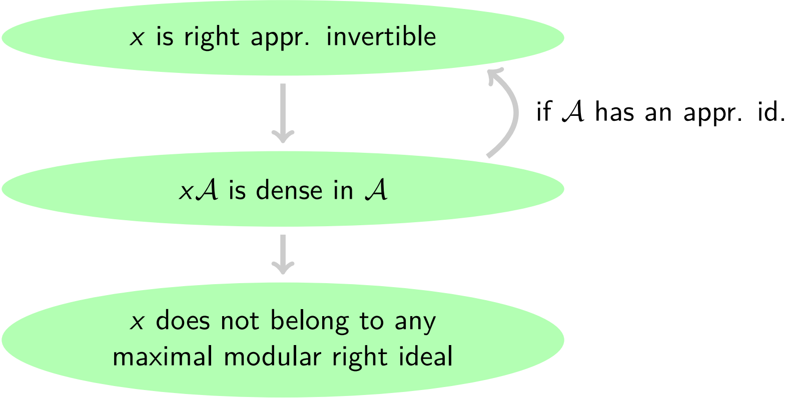
Idea of proof. Let $(e_j)_{j \in J}$ be an approximate identity of $\mathcal{A}$.
For every $j \in J$ and $\delta \in (0,1)$ choose $u_{j,\delta} \in \mathcal{A}$ such that

$$\|xu_{j,\delta} - e_j\| < \delta.$$

Then $(xu_{j,\delta})_{(j,\delta)\in J\times(0,1)}$ is an approximate identity in $\mathcal{A}$. □

Summary: non-commutative situation

Suppose that $\mathcal{A}$ is a normed algebra and $x \in \mathcal{A}$.



Problem 1

Let $\mathcal{A}$ is a non-unital normed algebra and $x \in \mathcal{A}$.
Suppose that x is approximately invertible from both sides, i.e. there exist nets $(u_j)_{j\in J}$ and $(v_k)_{k\in K}$ such that

$$(u_j x)_{j\in J} \quad \text{is appr.id.,} \qquad (x v_k)_{k\in K} \quad \text{is appr.id.}$$

Does there exists a net $(y_p)_{p\in P}$ such that $(x y_p)_{p\in P}$ and $(y_p x)_{p\in P}$ are approximate identities?

About the Gelfand transform of an approximate identity

Let $\mathcal{A}$ be a commutative Banach algebra ($\mathcal{A}$ may be non-unital).
Denote by $\mathcal{M}_{\mathcal{A}}$ the space of characters of $\mathcal{A}$.
For every $a \in \mathcal{A}$, denote by $\widehat{a}$ the Gelfand transform of a.

Proposition

Let $(e_j)_{j\in J}$ be an approximate identity in $\mathcal{A}$. Then for every $\varphi \in \mathcal{M}_{\mathcal{A}}$

$$\lim_{j\in J} \widehat{e}_j(\varphi) = 1.$$

Proof. Let $\varphi \in \mathcal{M}_{\mathcal{A}}$. Choose $a \in \mathcal{A}$ such that $\varphi(a) \neq 0$. Then

$$|\varphi(a)|\,|\widehat{e}_j(\varphi) - 1| = |\varphi(a)(\varphi(e_j) - 1)| = |\varphi(ae_j - a)| \leq \|ae_j - a\| \to 0.$$

Since $\varphi(a) \neq 0$, $\widehat{e}_j(\varphi) \to 1$. □

If $x\mathcal{A}$ is dense, then $\hat{x}$ does not vanish

Let $\mathcal{A}$ be a commutative Banach algebra.

Proposition

Let $x \in \mathcal{A}$ such that $x\mathcal{A}$ is dense in $\mathcal{A}$.
Then $\widehat{x}(\varphi) \neq 0$ for every $\varphi \in \mathcal{M}_{\mathcal{A}}$.

Proof. Let $\varphi \in \mathcal{M}_{\mathcal{A}}$ and $a \in \mathcal{A}$ such that $\varphi(a) = 1$.
Find $y \in \mathcal{A}$ such that $\|xy - a\| < 1$. Then

$$|\varphi(a - xy)| \leq \|a - xy\| < 1$$

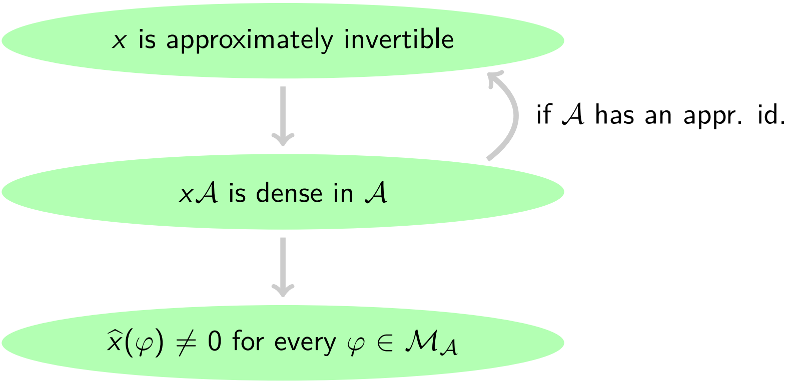
and

$$|\widehat{x}(\varphi)\widehat{y}(\varphi)| = |\varphi(xy)| \geq |\varphi(a)| - |\varphi(a - xy)| > 0,$$

which implies that $\widehat{x}(\varphi) \neq 0$. □

Summary: commutative situation

Suppose that $\mathcal{A}$ is a commutative Banach algebra and $x \in \mathcal{A}$.



Problem 2

Let $\mathcal{A}$ be a commutative Banach algebra
and $(e_j)_{j \in J}$ be an approximate identity in $\mathcal{A}$.

Does $(\widehat{e}_j)_{j \in J}$ converge to 1 uniformly on compacts?