Avram-Parter and Szegő theorems: convex test functions and counterexamples

Egor Maximenko

National Polytechnic Institute, Mexico City, Mexico

The 22nd International Workshop
on Operator Theory and its Applications
Seville, Spain, July 3–9, 2011

The results are joint with

Albrecht Böttcher (Chemnitz, Germany)

and Sergey M. Grudsky (Mexico City, Mexico).

Contents

Contents

Toeplitz matrices

Notation (Fourier coefficients of a function $a \in L^1([0, 2\pi])$)

$$a_k := \frac{1}{2\pi} \int_0^{2\pi} a(\theta) e^{-ik\theta} \, d\theta,$$

Definition (Toeplitz matrix of order n generated by $a \in L^1([0, 2\pi])$)

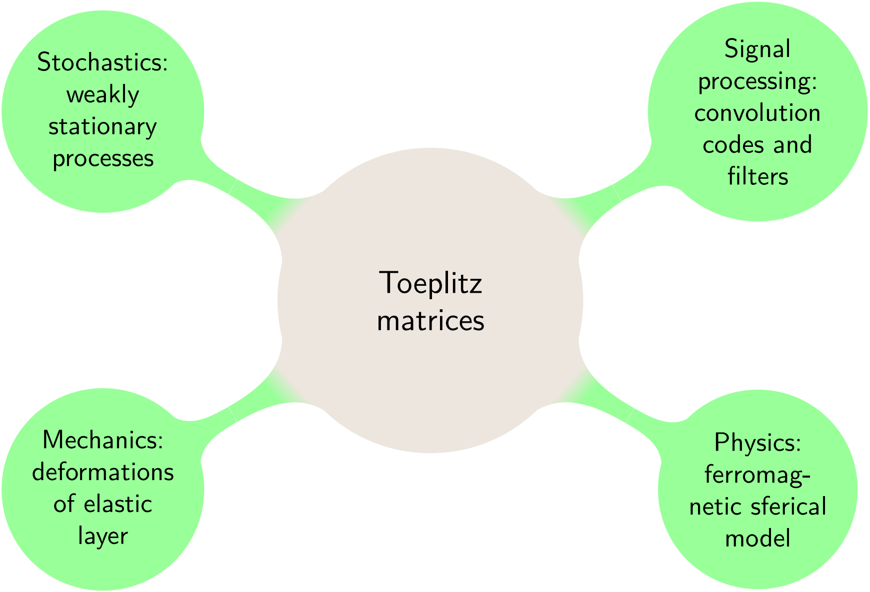
$$T_n(a) := [a_{j-k}]_{j,k=1}^n.$$

For $n = 4$,

$$T_4(a) = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_1 & a_0 & a_{-1} & a_{-2} \\ a_2 & a_1 & a_0 & a_{-1} \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix}$$

The function a is called the generating function or the generating symbol of the sequence of the matrices $T_n(a)$, $n = 1, 2, 3, \ldots$

Where do Toeplitz matrices appear?



Notation for the eigenvalues and singular values

Notation (eigenvalues of a self-adjoint matrix)

Given a self-adjoint matrix $A \in \mathbf{C}^{n\times n}$, denote by $\lambda_1(A), \ldots, \lambda_n(A)$ the eigenvalues of A counted according to their algebraic multiplicities. For example, we can enumerate them in the ascending order:

$$\lambda_1(A) \le \lambda_2(A) \le \ldots \le \lambda_n(A).$$

Notation (singular values of a matrix)

Given $A \in \mathbf{C}^{n\times n}$, denote by $s_1(A), \ldots, s_n(A)$ the singular values of A:

$$s_k(A) := \sqrt{\lambda_k(A^*A)}.$$

Tridiagonal real symmetric Toeplitz matrices

Consider a simple example:

$$a(\theta) = -e^{-i\theta} + 2 - e^{i\theta} = 2 - 2\cos\theta,$$

The Toeplitz matrices $T_n(a)$ generated by this symbol are real, symmetric, positive-definite and tridiagonal.

$$T_5(a) = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

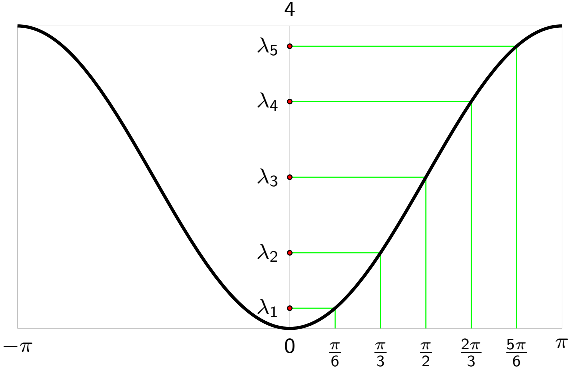
Since $T_n(a)$ are positive-definite, the singular values coincide with the corresponding eigenvalues:

$$s_k(T_n(a)) = \lambda_k(T_n(a)).$$

Eigenvalues of tridiagonal real symmetric Toeplitz matrices

In this example, $a(\theta) = 2 - 2\cos\theta$, there is a simple formula for $\lambda_k(T_n(a))$:

$$\lambda_k(T_n(a)) = a\left(\frac{k\pi}{n+1}\right) = 2 - 2\cos\frac{k\pi}{n+1}.$$



$n = 5$

Eigenvalues and the values of the generating function

In the previous example, the eigenvalues of $T_n(a)$ are equal to the values of the generating function at the uniformly distributed points $\dfrac{k\pi}{n+1}$:

$$\lambda_k(T_n(a)) = a\left(\frac{k\pi}{n+1}\right) \qquad (k = 1, 2, \ldots, n).$$

Hypothesis

For every real-valued, even and sufficiently smooth generating function a,

$$\lambda_k(T_n(a)) = a\left(\frac{k\pi}{n+1}\right) + \mathcal{O}\left(\frac{1}{n^2}\right).$$

In fact, the individual behavior of the eigenvalues of Toeplitz matrices is known only for few narrow classes of generating functions.

Contents

Averaged collective behavior of eigen- and singular values

It is easier to study the behavior of λ_j and s_j in the sense of distributions.

Associate the weight $\frac{1}{n}$ with each eigenvalue λ_j of the matrix $T_n(a)$, i.e., consider the Radon measure

$$F \quad \mapsto \quad \frac{1}{n}\sum_{i=1}^{n} F\Big(\lambda_j(T_n(a))\Big).$$

Here F is a continuous function called test function.

In a similar manner associate the weight $\frac{1}{n}$ with each singular value s_j:

$$F \quad \mapsto \quad \frac{1}{n}\sum_{i=1}^{n} F\Big(s_j(T_n(a))\Big).$$

In other words we consider the averaged F-sums of λ_j and s_j.

Szegő and Avram-Parter theorems

G. Szegő, 1920

Let $a \in L^\infty([0, 2\pi], \mathbf{R})$, $F \in C(\mathbf{R})$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F\Big(\lambda_k(T_n(a))\Big) = \frac{1}{2\pi} \int_0^{2\pi} F\Big(a(\theta)\Big)\, d\theta. \qquad \text{(S)}$$

S. V. Parter (1986), F. Avram (1988)

Let $a \in L^\infty([0, 2\pi], \mathbf{C})$, $F \in C(\mathbf{R}_+)$, $\mathbf{R}_+ := [0, +\infty)$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F\Big(s_k(T_n(a))\Big) = \frac{1}{2\pi} \int_0^{2\pi} F\Big(|a(\theta)|\Big)\, d\theta. \qquad \text{(AP)}$$

Szegő and Avram-Parter theorems

G. Szegő, 1920

Let $a \in L^\infty([0, 2\pi], \mathbf{R})$, $F \in C(\mathbf{R})$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F\Big(\lambda_k(T_n(a))\Big) = \frac{1}{2\pi} \int_0^{2\pi} F\Big(a(\theta)\Big)\, d\theta. \qquad \text{(S)}$$

S. V. Parter (1986), F. Avram (1988)

Let $a \in L^\infty([0, 2\pi], \mathbf{C})$, $F \in C(\mathbf{R}_+)$, $\mathbf{R}_+ := [0, +\infty)$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F\Big(s_k(T_n(a))\Big) = \frac{1}{2\pi} \int_0^{2\pi} F\Big(|a(\theta)|\Big)\, d\theta. \qquad \text{(AP)}$$

Question

Do the formulas (AP) and (S) hold for non-bounded generating functions?

Positive test functions, infinite values in (S) and (AP)

The left-hand and right-hand sides of (S) and (AP) are linear in F:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F(\lambda_k(T_n(a))) = \frac{1}{2\pi} \int_0^{2\pi} F(a(\theta))\, d\theta. \qquad \text{(S)}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F\Big(s_k(T_n(a))\Big) = \frac{1}{2\pi} \int_0^{2\pi} F\Big(|a(\theta)|\Big)\, d\theta. \qquad \text{(AP)}$$

Since every **C**-valued function is a lin. comb. of four $\mathbf{R}_+$-valued functions, without loss of generality consider only positive test functions:

$$F(x) \geq 0 \qquad \forall x.$$

Working with positive test functions we may and shall admit infinite values in both sides of (AP) and (S).

Notation: classes of "good" test functions

$\mathcal{APT}$:= the test functions that fulfill (AP) for all integrable symbols

$$\mathcal{APT} := \Big\{ F \in C(\mathbf{R}_+, \mathbf{R}_+) : \quad (\mathrm{AP}) \text{ holds for this test function } F \text{ and all symbols } a \in L^1([0,2\pi], \mathbf{C}) \Big\}.$$

$\mathcal{ST}$:= the test functions that fulfill (S) for all integrable symbols

$$\mathcal{ST} := \Big\{ F \in C(\mathbf{R}, \mathbf{R}_+) : \quad (\mathrm{S}) \text{ holds for this test function } F \text{ and all symbols } a \in L^1([0,2\pi], \mathbf{R}) \Big\}.$$

Goal

Determine what functions belong to the classes $\mathcal{APT}$ and $\mathcal{ST}$.

Here we find wide sufficient conditions and some counterexamples.

Known results for non-bounded symbols

1997 E. E. Tyrtyshnikov and N. L. Zamarashkin:
$a \in L^1, \quad F \in C_0.$

Known results for non-bounded symbols

1997 E. E. Tyrtyshnikov and N. L. Zamarashkin:
$a \in L^1, \quad F \in C_0$.

1998 P. Tilli:
$a \in L^1, \quad F$ is uniformly continuous (simple proof!)

Known results for non-bounded symbols

1997 E. E. Tyrtyshnikov and N. L. Zamarashkin:
$a \in L^1, \quad F \in C_0.$

1998 P. Tilli:
$a \in L^1, \quad F$ is uniformly continuous (simple proof!)

2002 S. Serra Capizzano:
(AP) for $a \in L^p, \quad F(s) = \mathcal{O}(s^p)$ as $s \to +\infty$.

Known results for non-bounded symbols

1997 E. E. Tyrtyshnikov and N. L. Zamarashkin:
$a \in L^1, \quad F \in C_0.$

1998 P. Tilli:
$a \in L^1, \quad F$ is uniformly continuous (simple proof!)

2002 S. Serra Capizzano:
(AP) for $a \in L^p, \quad F(s) = \mathcal{O}(s^p)$ as $s \to +\infty$.

2008 A. Böttcher, S. M. Grudsky, M. Schwartz:
For all $a \in L^1$, $F \in C$ the following inequality holds:

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F\Big(s_k(T_n(a))\Big) \geq \frac{1}{2\pi} \int_0^{2\pi} F\Big(|a(\theta)|\Big)\, d\theta.$$

Known results for non-bounded symbols

1997 E. E. Tyrtyshnikov and N. L. Zamarashkin:
$a \in L^1, \quad F \in C_0.$

1998 P. Tilli:
$a \in L^1, \quad F$ is uniformly continuous (simple proof!)

2002 S. Serra Capizzano:
(AP) for $a \in L^p, \quad F(s) = \mathcal{O}(s^p)$ as $s \to +\infty$.

2008 A. Böttcher, S. M. Grudsky, M. Schwartz:
For all $a \in L^1$, $F \in C$ the following inequality holds:

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F\Big(s_k(T_n(a))\Big) \geq \frac{1}{2\pi} \int_0^{2\pi} F\Big(|a(\theta)|\Big)\, d\theta.$$

BGS also constructed a counterexample for (AP) and (S).
In that counterexample $a \in L^1$, F is continuous, but not monotonous.

Contents

Counterexample with increasing test function

Theorem 1

There exists a generating function $a \in L^1([0, 2\pi], \mathbf{R}_+)$ *and a strictly increasing test function* $F \in C(\mathbf{R}_+, \mathbf{R}_+)$ *such that*

$$\int_0^{2\pi} F\Big(a(\theta)\Big)\, d\theta < +\infty,$$

but

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F\Big(s_k(T_n(a))\Big) = +\infty.$$

So, for these a and F formulas (AP) and (S) do not hold.
In particular, $F \notin \mathcal{APT}$.

Counterexample with increasing test function

Construction of the generating function a

The generating function is piecewise-constant:

$$a(\theta) := \begin{cases} b_k, & \theta \in [(1-\delta_k)\beta_k, \beta_k] \qquad (k \geq 1); \\ 0, & \text{for others } \theta. \end{cases}$$

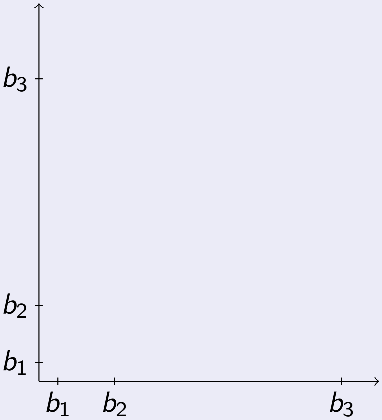
The values b_k, β_k and δ_k are chosen in the following manner:

$$b_k := 2^{k^2}, \qquad \beta_k := \frac{2\pi}{b_k}, \qquad \delta_k := \frac{1}{k(k+1)}.$$

Then $\quad \|a\|_{L^1} = 1 \quad$ but $\quad \|a\|_\infty = +\infty \quad$ and $\quad \|T_n(a)\| \to +\infty.$

Counterexample with increasing test function

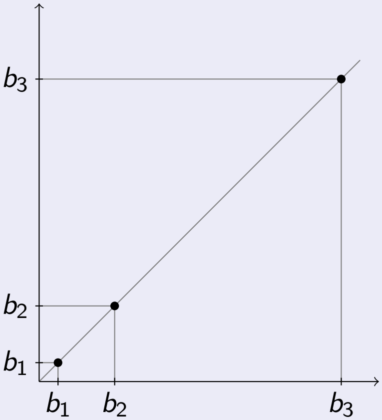
Construction of the test function F



F has the following properties:

Counterexample with increasing test function

Construction of the test function F

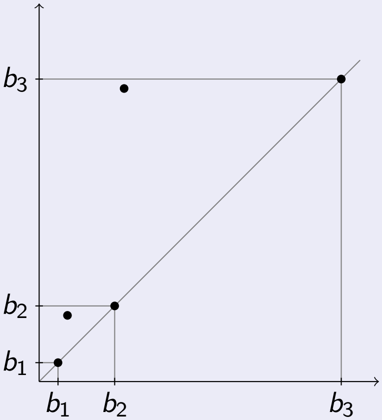


F has the following properties:

$$F(b_k) = b_k,$$

Counterexample with increasing test function

Construction of the test function F



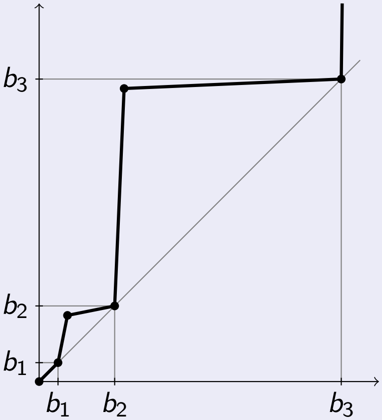
F has the following properties:

$$F(b_k) = b_k,$$

$$F(b_k + 1) = b_{k+1} - 1,$$

Counterexample with increasing test function

Construction of the test function F



F has the following properties:

$$F(b_k) = b_k,$$

$$F(b_k + 1) = b_{k+1} - 1,$$

F strictly increases,

F is continuous.

Another counterexample with increasing test function

Theorem 2

There exists a generating function $a \in L^1([0, 2\pi], \mathbf{R}_+)$ and a strictly increasing test function $F \in C(\mathbf{R}_+, \mathbf{R}_+)$ such that

$$+\infty \quad > \quad \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} F\Big(s_k(T_n(a))\Big) \quad > \quad \frac{1}{2\pi} \int_0^{2\pi} F\Big(a(\theta)\Big)\, d\theta.$$

So, for these a and F formulas (AP) and (S) fail,
but both sides of them are finite.

Proof.

A slight modification of the previous construction. □

Contents

Terminology: increasing and convex functions

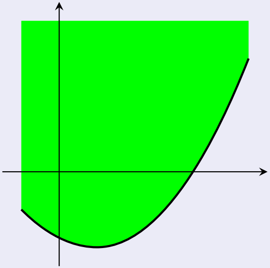
Increasing function: in the non-strict sense

$$\forall x_1, x_2 \in \mathrm{Dom}(f) \qquad \Big(x_1 \le x_2 \quad \Longrightarrow \quad f(x_1) \le f(x_2) \Big).$$

Convex function: in the $\smile$ sense

f is convex $\iff$

epigraph(f) is a convex set



Inequality for the singular values of Toeplitz matrices

Key result

Theorem 3

Let $a \in L^1([0, 2\pi], \mathbf{C})$ *and* $\Phi\colon \mathbf{R}_+ \to \mathbf{R}_+$ *be an increasing convex function. Then for every* $n \in \{1, 2, 3, \ldots\}$,

$$\frac{1}{n} \sum_{k=1}^{n} \Phi\Big(s_k(T_n(a))\Big) \leq \frac{1}{2\pi} \int_0^{2\pi} \Phi\Big(|a(\theta)|\Big)\, d\theta.$$

Inequality for the singular values of Toeplitz matrices

Key result

Theorem 3

Let $a \in L^1([0, 2\pi], \mathbf{C})$ *and* $\Phi\colon \mathbf{R}_+ \to \mathbf{R}_+$ *be an increasing convex function. Then for every* $n \in \{1, 2, 3, \ldots\}$,

$$\frac{1}{n}\sum_{k=1}^{n} \Phi\Big(s_k(T_n(a))\Big) \le \frac{1}{2\pi}\int_0^{2\pi} \Phi\Big(|a(\theta)|\Big)\, d\theta.$$

Note

For $\Phi(s) = s^p$ this inequality was proved
by S. Serra Capizzano and P. Tilli (2002).

Inequality for the singular values of Toeplitz matrices

Key result

Theorem 3

Let $a \in L^1([0,2\pi],\mathbf{C})$ *and* $\Phi\colon \mathbf{R}_+ \to \mathbf{R}_+$ *be an increasing convex function. Then for every* $n \in \{1,2,3,\ldots\}$,

$$\frac{1}{n}\sum_{k=1}^{n} \Phi\Big(s_k(T_n(a))\Big) \le \frac{1}{2\pi}\int_0^{2\pi} \Phi\Big(|a(\theta)|\Big)\,d\theta.$$

Note

For $\Phi(s) = s^p$ this inequality was proved by S. Serra Capizzano and P. Tilli (2002).

Ideas of the proof

The proof is based on Jensen's inequality for convex functions and on Sing's and Thompson's inequality for the diagonal and singular values.

Avram-Parter formula for convex test functions

The class $\mathcal{APT}$ contains all convex functions $\mathbf{R}_+ \to \mathbf{R}_+$.

Avram-Parter formula for convex test functions

The class $\mathcal{APT}$ contains all convex functions $\mathbf{R}_+ \to \mathbf{R}_+$.

Using some ideas from [Serra Capizzano, 2002] we generalize this result to the test function that are dominated by convex functions:

Theorem 4

Let

- $a \in L^1([0, 2\pi], \mathbf{C})$,
- $\Phi\colon \mathbf{R}_+ \to \mathbf{R}_+$ *be a convex function,*
- $\Phi \circ |a| \in L^1$,
- $F \in C(\mathbf{R}_+, \mathbf{R}_+)$ *and* $F(x) \le \Phi(x)$ *for* $x > x_0$.

Then for these functions a and F formula (AP) *holds.*

Szegő formula for convex test functions

The class $\mathcal{ST}$ contains all convex functions (acting from $\mathbf{R}$ to $\mathbf{R}_+$).

Szegő formula for convex test functions

The class $\mathcal{ST}$ contains all convex functions (acting from $\mathbf{R}$ to $\mathbf{R}_+$).

And a more general version for the test functions
whose positive and negative parts are dominated by convex functions:

Theorem 5

Let

- $a \in L^1([0, 2\pi], \mathbf{R})$,
- $\Phi_\pm : \mathbf{R}_+ \to \mathbf{R}_+$ *are increasing and convex, and* $\Phi_+(0) = \Phi_-(0)$,
- $\Phi_\pm \circ a_\pm \in L^1$, *where*

$$a_+(\theta) = \max(a(\theta), 0), \qquad a_-(\theta) = \max(-a(\theta), 0),$$

- $F \in C(\mathbf{R}, \mathbf{R}_+)$, $F(x) \le \Phi_+(x)$ *and* $F(-x) \le \Phi_-(x)$ *for* $x \ge x_0$.

Then for these functions a *and* F *formula* (S) *holds.*

Main references I

F. Avram,
On bilinear forms in Gaussian random variables and Toeplitz matrices,
Probab. Theory Related Fields **79** (1988), 37–45.

A. Böttcher and B. Silbermann,
Introduction to Large Truncated Toeplitz Matrices, 1999.

A. Böttcher, S. M. Grudsky, and M. Schwartz,
Some problems concerning the test functions in the Szegö and Avram-Parter theorems,
Operator Theory: Advances and Applications, **187** (2008), 81–93.

U. Grenander and G. Szegö,
Toeplitz Forms and Their Applications, 1958.

S. V. Parter,
On the distribution of the singular values of Toeplitz matrices,
Linear Algebra Appl. **80** (1986), 115–130.

Main references II

S. Serra Capizzano,
Test functions, growth conditions and Toeplitz matrices,
Rend. Circ. Mat. Palermo, Ser. II, Suppl. **68** (2002), 791–795.

S. Serra Capizzano and P. Tilli,
On unitarily invariant norms of matrix-valued linear positive operators,
J. Inequal. Appl. **7** (2002), 309–330.

P. Tilli,
A note on the spectral distribution of Toeplitz matrices,
Linear and Multilinear Algebra **45** (1998), 147–159.

E. E. Tyrtyshnikov and N. L. Zamarashkin,
Spectra of multilevel Toeplitz matrices: advanced theory via simple matrix relationships,
Linear Algebra Appl. **270** (1998), 15–27.