On the asymptotics of the eigenvectors of certain Hermitian Toeplitz matrices

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Notation: Hermitian Toeplitz matrices

Let $a \in L^1(\mathbf{T}, \mathbf{R})$.
Denote by g the corresponding 2π-periodical function on $\mathbf{R}$:

$$g(x) := a(e^{ix}), \qquad x \in \mathbf{R}.$$

Let a_k $(k \in \mathbf{Z})$ be the Fourier coefficients of a:

$$a_k := \int_{\mathbf{T}} a(t) t^{-k} \, d\mu_{\mathbf{T}}(t) = \frac{1}{2\pi} \int_0^{2\pi} g(x) e^{-ikx} \, dx.$$

Consider Toeplitz matrices $T_n(a)$, $n = 1, 2, 3, \ldots$, generated by a:

$$T_n(a) := (a_{j-k})_{j,k=1}^n = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \ldots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \ldots & a_{-n+2} \\ a_2 & a_1 & a_0 & \ldots & a_{-n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_0 \end{pmatrix}.$$

Notation: eigenvalues and eigenvectors

Denote by $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ the eigenvalues of $T_n(a)$ in the increasing order:

$$\lambda_1^{(n)} \le \lambda_2^{(n)} \le \ldots \le \lambda_n^{(n)},$$

and by $v_1^{(n)}, \ldots, v_n^{(n)}$ the corresponding normalized eigenvectors:

$$T_n(a) v_k^{(n)} = \lambda_k^{(n)} v_k^{(n)}, \qquad \|v_k^{(n)}\|_2 = 1.$$

Under some assumptions on the symbol, the eigenvalues are simple. Therefore every normalized eigenvector is defined uniquely up to unitary multiplier τ, $|\tau| = 1$.

"Minimal distance" between normalized vectors

$$\varrho(u, v) := \min_{|\tau|=1} \|\tau u - v\|_2 = \left\| \frac{|\langle u, v \rangle|}{\langle u, v \rangle} u - v \right\|_2 .$$

Tridiagonal real symmetric Toeplitz matrices

As a model example, consider the following three-term symbol:

$$a(t) = -t + 2 - t^{-1}$$

In this case the function $g(x) := a(e^{ix})$ is

$$g(x) = 4\sin^2\frac{x}{2}.$$

The corresponding Toeplitz matices are tridiagonal. For instance,

$$T_5(-t+2-t^{-1}) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The eigenvalues and eigenvectors of these matrices are well known.

Eigenvalues and -vectors of tridiagonal Toeplitz matrices

The k-st eigenvalue of $T_n(-t+2-t^{-1})$ is $4\sin^2\frac{k\pi}{2(n+1)}$.

Denote the corresponding eigenvector by $x_k^{(n)}$:

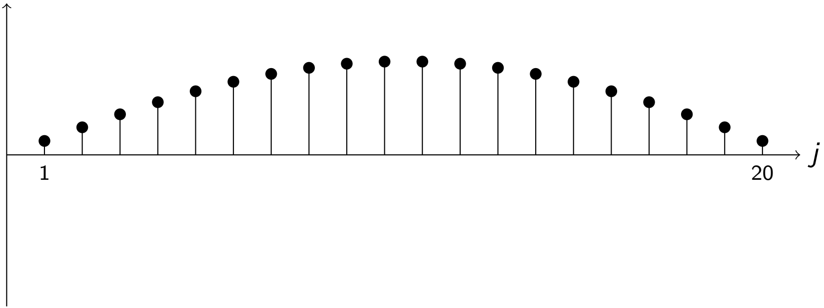
$$x_k^{(n)} = \left(\sqrt{\frac{2}{\pi}}\sin\frac{kj\pi}{n+1}\right)_{j=1}^{n}.$$

Eigenvalues and -vectors of tridiagonal Toeplitz matrices

The k-st eigenvalue of $T_n(-t+2-t^{-1})$ is $\quad 4\sin^2\frac{k\pi}{2(n+1)}$.

Denote the corresponding eigenvector by $x_k^{(n)}$:

$$x_1^{(20)} = \left(\sqrt{\frac{2}{\pi}}\sin\frac{j\pi}{21}\right)_{j=1}^{20}.$$

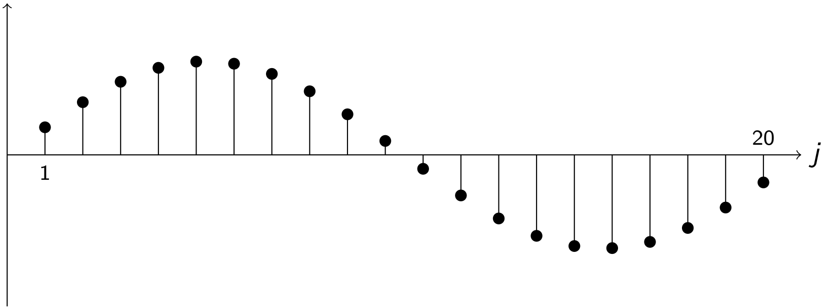


Eigenvalues and -vectors of tridiagonal Toeplitz matrices

The k-st eigenvalue of $T_n(-t+2-t^{-1})$ is $\quad 4\sin^2\frac{k\pi}{2(n+1)}$.

Denote the corresponding eigenvector by $x_k^{(n)}$:

$$x_2^{(20)} = \left(\sqrt{\frac{2}{\pi}}\sin\frac{2j\pi}{21}\right)_{j=1}^{20}.$$



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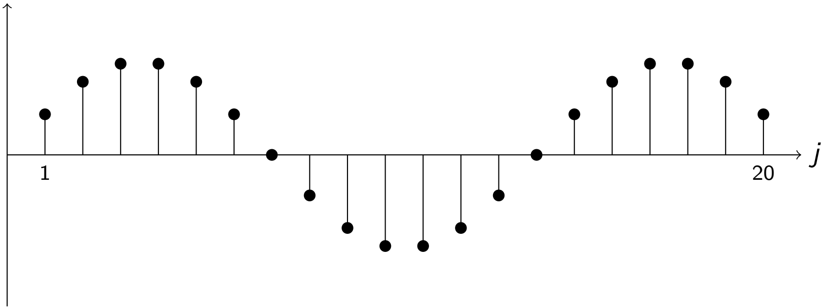
The k-st eigenvalue of $T_n(-t+2-t^{-1})$ is $4\sin^2\frac{k\pi}{2(n+1)}$.

Denote the corresponding eigenvector by $x_k^{(n)}$:

$$x_3^{(20)} = \left(\sqrt{\frac{2}{\pi}}\sin\frac{3j\pi}{21}\right)_{j=1}^{20}.$$

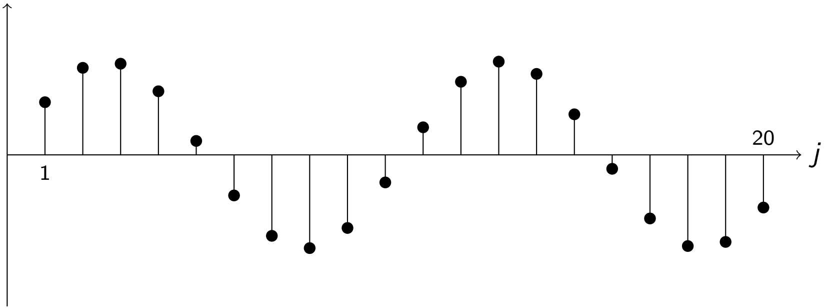


Eigenvalues and -vectors of tridiagonal Toeplitz matrices

The k-st eigenvalue of $T_n(-t+2-t^{-1})$ is $4\sin^2\dfrac{k\pi}{2(n+1)}$.

Denote the corresponding eigenvector by $x_k^{(n)}$:

$$x_4^{(20)} = \left(\sqrt{\frac{2}{\pi}}\sin\frac{4j\pi}{21}\right)_{j=1}^{20}.$$

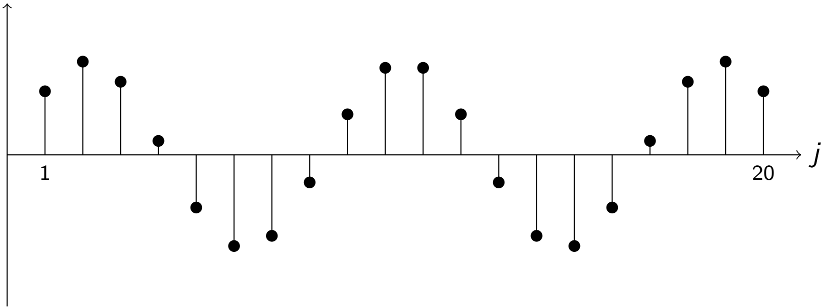


Eigenvalues and -vectors of tridiagonal Toeplitz matrices

The k-st eigenvalue of $T_n(-t+2-t^{-1})$ is $\quad 4\sin^2\frac{k\pi}{2(n+1)}$.

Denote the corresponding eigenvector by $x_k^{(n)}$:

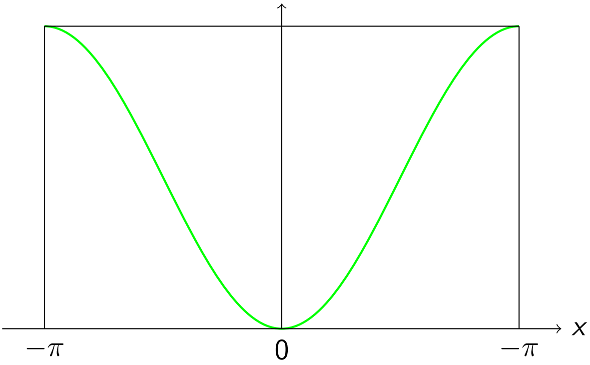
$$x_5^{(20)} = \left(\sqrt{\frac{2}{\pi}}\sin\frac{5j\pi}{21}\right)_{j=1}^{20}.$$



Important properties of the model generating function

The function $g(x) = 4\sin^2\frac{x}{2}$ has the following properties:

- It has only one minimum (on $[-\pi,\pi]$);
- The minimal value of g is reached at the point 0;
- $g''(0) > 0$ ("second-order" minimum).



Some known results on the extreme eigenvalues

M. Kac, W. L. Murdock, G. Szegö, 1953. Let $a \in C(\mathbf{T}, \mathbf{R})$, $g(x)$ reaches its minimal value only at $x = 0$, $g''(0) > 0$, $g \in C^2$ near 0. Then $\forall k$

$$\lambda_k^{(n)} = g(0) + \frac{g''(0)\pi^2}{2} \cdot \frac{k^2}{(n+1)^2} + o\left(\frac{1}{(n+1)^2}\right) \qquad \text{as} \quad n \to \infty.$$

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H. Widom, 1958. Assume additionally that g is even and $g \in C^4$ near 0. Then $\forall k$

$$\lambda_k^{(n)} = g(0) + \frac{g''(0)\pi^2}{2} \left(\frac{k^2}{(n+1)^2} + \frac{w_0 k^2}{(n+1)^3}\right) + o\left(\frac{1}{(n+1)^3}\right).$$

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S. Serra Capizzano, 1997. Let $a \in L^1(\mathbf{T}, \mathbf{R})$ and $\exists c_1, c_2 > 0$ such that

$$c_1 x^2 \le g(x) - \operatorname{ess\,inf}(g) \le c_2 x^2.$$

Then $\exists c_3, c_4 > 0$ such that $\quad \frac{c_3}{n^2} \le \lambda_1^{(n)} - \operatorname{ess\,inf}(g) \le \frac{c_4}{n^2}.$

Rest of the talk: two new results about the eigenvectors

Main term of the first eigenvectors (integrable symbols)

Suppose that:

- the generating function is real-valued and integrable on $[-\pi, \pi]$;
- reaches its minimal value only at the point 0;
- this minimum is of the second order.

Then the first eigenvectors of the Toeplitz matrices are close to the eigenvectors of the tridiagonal symmetric Toeplitz matrices.

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Precise asymptotics of all eigenvectors (polynomial symbols)

Suppose that the symbol is a real-valued Laurent polynomial, has a unique minimum, a unique maximum, and is monotone between the minimum and maximum.

In this case, exponentially precise asymptotical formulas are obtained for all the eigenvectors (not only for the first ones).

Contents

Result

Theorem

Let $a \in L^1(\mathbf{T}, \mathbf{R})$*,* $m \in \mathbf{R}$*,* $A > 0$*,* $\delta_0 > 0$*,* $\eta > 0$*,*

- $g(x) = m + Ax^2 + x^2\xi(x)$*,*
  where $\lim_{x \to 0} \xi(x) = 0$ *and* $|\xi(x)| < \frac{A}{2}$ *if* $|x| < \delta_0$*.*
- $g(x) \geq m + \eta$ *if* $\delta_0 \leq |x| \leq \pi$*.*

Then for each fixed $k = 1, 2, 3, \ldots$ *the following asymptotical formulas hold as* $n \to \infty$*:*

$$\lambda_k^{(n)} = m + A\frac{k^2\pi^2}{(n+1)^2} + o\left(\frac{1}{(n+1)^2}\right),$$

$$\varrho(v_k^{(n)}, x_k^{(n)}) \to 0.$$

Proof: estimates for the “Rayleigh quotients” $\langle T_n(a)x_k^{(n)}, x_k^{(n)}\rangle$.

Example with nonbounded integrable symbol

$$a(e^{ix}) = \left|1 + e^{ix}\right|^{-2\alpha} = 2^{-2\alpha}\left(\cos\frac{x}{2}\right)^{-2\alpha}.$$

In this example $g(x) = m + Ax^2 + O(x^4)$ as $x \to 0$,
where $m = 2^{-2\alpha}$, $A = 4\alpha 2^{-2\alpha-2}$, and $g(x) \to +\infty$ as $x \to \pm\pi$.

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where $m = 2^{-2\alpha}$, $A = 4\alpha 2^{-2\alpha-2}$, and $g(x) \to +\infty$ as $x \to \pm\pi$.

Consider the error terms in the asymptotical formulas for $\lambda_k^{(n)}$ and $v_k^{(n)}$:

$$X_k^{(n)} := \lambda_k^{(n)} - m - \frac{Ak^2\pi^2}{(n+1)^2}, \qquad Y_k^{(n)} := \varrho\left(v_k^{(n)}, x_k^{(n)}\right).$$

Our method gives the following upper estimates of these errors:

$$X_k^{(n)} = O\left(\frac{k^3}{n^3}\right), \qquad Y_k^{(n)} = O\left(\frac{\beta_k}{\sqrt{n}}\right).$$

Computations (for $\alpha = 1/4$) show that

$$X_k^{(n)} = O\left(\frac{k^2}{n^3}\right), \qquad Y_k^{(n)} = O\left(\frac{k}{n}\right).$$

Contents

Instruments: formulas by Widom and Trench

Notation: complex roots of a polynomial symbol

Let a be a Laurent polinomial: $a(z) = \sum_{k=-q}^{p} a_k z^k \quad (a_p \neq 0,\ a_{-q} \neq 0)$.

Denote by $z_1(a), \ldots, z_{p+q}(a)$ the roots of the polynomial $z^q a(z)$.

Formulas by Widom and Trench

H. Widom (1958):

formulas for the determinants $\det T_n(a)$ in terms of $z_k(a)$.

W. F. Trench (1985):

formulas for the elements of inverse matrices $T_n^{-1}(a)$ in terms of $z_k(a)$.

These formulas are especially simple when the roots $z_1(a), \ldots, z_{p+q}(a)$ are pairwise distinct.

Assumptions on the symbol

- The symbol is a non-constant Laurent polynomial:

$$a(t) = \sum_{k=-r}^{r} a_k t^k, \qquad r \geq 1, \quad a_r \neq 0, \quad a_{-r} \neq 0.$$

- a is real-valued on $\mathbf{T}$ $\quad (\Leftrightarrow \quad \overline{a_k} = a_{-k} \quad$ for all $k)$.
- $a(\mathbf{T}) = [0, M]$.
- $g(0) = 0, \quad g(\varphi_0) = M \quad$ for some $\varphi_0 \in (0, 2\pi)$.
  Here $g(x) = a(e^{ix})$.
- g is strictly increasing on $[0, \varphi_0]$, strictly decreasing on $[\varphi_0, 2\pi]$,
  $g''(0) > 0$ and $g''(\varphi_0) < 0$.
- Technical assumption: for each $\lambda \in (0, M)$
  the roots of $a(z) - \lambda$ lying in $\mathbf{C} \setminus \mathbf{T}$ are pairwise distinct.

Functions φ and θ

For every $\lambda \in (0, M)$, the roots of $a(z) - \lambda$ can be written as

$$u_1(\lambda), \ldots, u_{r-1}(\lambda), \qquad e^{i\varphi_1(\lambda)},\ e^{i\varphi_2(\lambda)}, \qquad \frac{1}{\overline{u_1(\lambda)}}, \ldots, \frac{1}{\overline{u_{r-1}(\lambda)}},$$

where $\quad 0 < \varphi_1(\lambda) < \varphi_0, \quad \varphi_0 - 2\pi < \varphi_2(\lambda) < 0, \quad |u_k(\lambda)| \geq 1 + \delta_0.$

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$$u_1(\lambda), \ldots, u_{r-1}(\lambda), \qquad e^{i\varphi_1(\lambda)},\ e^{i\varphi_2(\lambda)}, \qquad \frac{1}{\overline{u_1(\lambda)}}, \ldots, \frac{1}{\overline{u_{r-1}(\lambda)}},$$

where $\quad 0 < \varphi_1(\lambda) < \varphi_0, \quad \varphi_0 - 2\pi < \varphi_2(\lambda) < 0, \quad |u_k(\lambda)| \geq 1 + \delta_0.$

$$\varphi(\lambda) := \frac{\varphi_1(\lambda) - \varphi_2(\lambda)}{2} = \frac{1}{2}\,\mathrm{mes}\{x \in [0, 2\pi) \colon g(x) \leq \lambda\}.$$

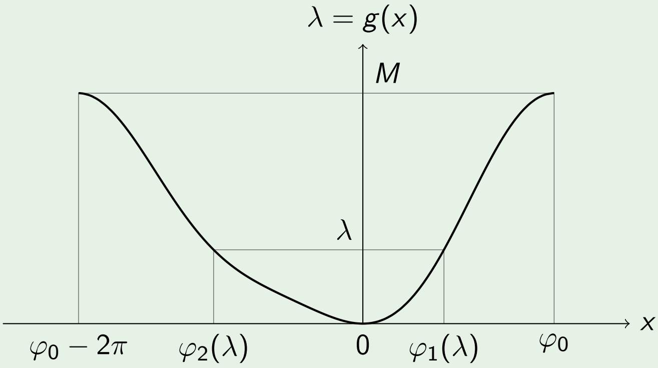
$$h_\lambda(z) := \prod_{k=1}^{r-1} \left(1 - \frac{z}{u_k(z)}\right).$$

$$\theta(\lambda) := \text{the continuous argument of } \frac{h_\lambda(e^{i\varphi_1(\lambda)})}{h_\lambda(e^{i\varphi_2(\lambda)})} \text{ satisfying } \theta(0) = 0.$$

Function φ

$$\varphi(\lambda) := \frac{\varphi_1(\lambda) - \varphi_2(\lambda)}{2} = \frac{1}{2}\,\mathrm{mes}\{x \in [0, 2\pi) \colon g(x) \le \lambda\}.$$

Example $\left(g(x) = 16\sin^2\frac{x}{2} + 16\sin^3\frac{x}{2}\cos\frac{x}{2}\right)$



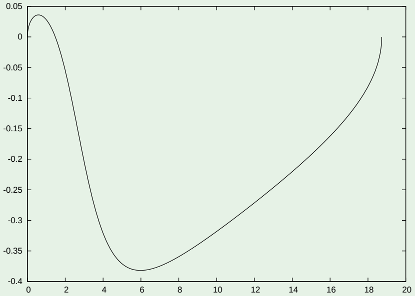
Function θ

θ is defined as the continuous argument of the function

$$\lambda \mapsto h_\lambda(e^{i\varphi_1(\lambda)})/h_\lambda(e^{i\varphi_2(\lambda)})$$

satisfying $\theta(0) = 0$.

Example (graph of $\theta(\lambda)$ for $g(x) = 16\sin^2\frac{x}{2} + 16\sin^3\frac{x}{2}\cos\frac{x}{2}$)



Asymptotical equation for the eigenvalues

Theorem

The solution $\lambda_{k,*}^{(n)}$ *of the equation*

$$(n+1)\varphi(\lambda) + \theta(\lambda) = k\pi \tag{$*$}$$

is exponentially close to $\lambda_k^{(n)}$ *as* $n \to \infty$*:*

$$|\lambda_k^{(n)} - \lambda_{k,*}^{(n)}| \leq Ce^{-\delta n}.$$

The solution of $(*)$ can be computed using the fixed point method:

$$\lambda_{k,0}^{(n)} := \varphi^{-1}\left(\frac{k\pi}{n+1}\right), \qquad \lambda_{k,j}^{(n)} := \varphi^{-1}\left(\frac{k\pi - \theta(\lambda_{k,j-1}^{(n)})}{n+1}\right).$$

Formulas for the eigenvectors (symmetric case)

Introduce the vectors $y_k^{(n)}$ with the following coordinates:

$$y_{k,m}^{(n)} := \sin\left(m\varphi(\lambda) + \frac{\theta(\lambda)}{2}\right) - \sum_{j=1}^{r-1} Q_j(\lambda)\left(\frac{1}{u_j(\lambda)^m} + \frac{(-1)^{k+1}}{u_j(\lambda)^{n+1-m}}\right),$$

$$\text{where} \quad Q_j(\lambda) = \frac{|h_\lambda(e^{i\varphi(\lambda)})| \sin\varphi(\lambda)}{(u_j(\lambda) - e^{i\varphi(\lambda)})(u_j(\lambda) - e^{i\varphi(\lambda)})h'_\lambda(u_j(\lambda))}, \quad \lambda = \lambda_k^{(n)}.$$

Let $w_k^{(n)}$ be the normalized vector $y_k^{(n)}$.

Theorem

$$\varrho(v_k^{(n)}, w_k^{(n)}) \le Ce^{-n\delta},$$

where C and δ depend only on the symbol.

In the nonsymmetric case the formulas for $y_k^{(n)}$ are a little more complicated.

Idea of the proof

Denote by $\operatorname{adj}(A)$ the matrix of the cofactors of a matrix A.
If λ is an eigenvalue of A, then

$$(A-\lambda I)\operatorname{adj}(A-\lambda I)=\det(A-\lambda I)I=0.$$

So, every non-zero column of the matrix $\operatorname{adj}(A-\lambda I)$ is an eigenvector of A corresponding to the eigenvalue λ.

We use the formulas by Trench to compute the elements of the matrix

$$\operatorname{adj}(T_n(a-\lambda_k^{(n)})).$$

Then we omit the exponentially small terms and arrive to $y_k^{(n)}$.

Formulas for the eigenvectors are exponentially precise

Consider the uniform errors:

$$\Delta_*^{(n)} := \max_{1\le k\le n} |\lambda_{k,*}^{(n)} - \lambda_k|; \qquad \Delta_v^{(n)} := \max_{1\le k\le n} \varrho(v_k^{(n)}, w_k^{(n)});$$

$$\Delta_r^{(n)} := \max_{1\le k\le n} \| T_n(a) w_k^{(n)} - \lambda_{k,*}^{(n)} w_k^{(n)} \|.$$

The theorems say that $\Delta_*^{(n)}$ and $\Delta_v^{(n)}$ are $O(e^{-\delta n})$.

Example ($g(x) = 4\sin^2\frac{x}{2} + 16\sin^4\frac{x}{2}$)

	$n = 10$	$n = 20$	$n = 50$	$n = 100$	$n = 150$
$\Delta_*^{(n)}$	$5.4 \cdot 10^{-7}$	$1.1 \cdot 10^{-11}$	$5.2 \cdot 10^{-25}$	$1.7 \cdot 10^{-46}$	$9.6 \cdot 10^{-68}$
$\Delta_v^{(n)}$	$2.0 \cdot 10^{-6}$	$1.1 \cdot 10^{-10}$	$2.0 \cdot 10^{-23}$	$1.9 \cdot 10^{-44}$	$2.0 \cdot 10^{-65}$
$\Delta_r^{(n)}$	$8.0 \cdot 10^{-6}$	$2.7 \cdot 10^{-10}$	$3.4 \cdot 10^{-23}$	$2.2 \cdot 10^{-44}$	$1.9 \cdot 10^{-65}$

First eigenvectors (symmetric case)

Corollary

$$\varrho\left(v_k^{(n)}, x_k^{(n)}\right) \leq \frac{Ck}{n+1},$$

where $x_k^{(n)}$ are the eigenvectors of tridiagonal symmetric Toeplitz matrices, and the constant C depends only on the symbol.

Example ($g(x) = 4\sin^2\frac{x}{2} + 16\sin^4\frac{x}{2}$)

The values of $\varrho\left(v_k^{(n)}, x_k^{(n)}\right)$ multiplied by $\frac{n+1}{k}$:

	$n = 100$	$n = 1000$	$n = 10000$
$k = 1$	1.212	1.273	1.279
$k = 2$	1.083	1.155	1.162
$k = 3$	1.052	1.132	1.139