Lower triangular sums

Objectives. Learn to write lower triangular sums in various manners: by rows, by columns and by diagonals.

Requirements. Experience of working with notation $\sum$, experience of programming (especially in some language that deals with matrices and starts indices by 1).

Lower triangular summing by rows

1. Consider the sum corresponding to the picture. First we write all summands in the explicit form (joining summands by rows):

   \[ S = (a_{1,1}) + (a_{2,1} + a_{2,2}) + ( ) + ( ). \]

   Now we rewrite each one of the little sums using the symbol $\sum$:

   \[ S = \sum_{k=1}^{1} a_{1,k} + \sum_{k=1}^{2} a_{2,k} + \sum + \sum . \]

   All summands have the same form, and we can write $S$ as a double sum. Write correctly the limits of the inner sum:

   \[ S = \sum_{j=1}^{4} \sum_{k=1}^{4} a_{j,k}. \]
2. Program (summing by rows the entries of a lower triangular matrix). Write a simple program that creates a pseudorandom lower triangular matrix and sums all its entries row by row. Here is a draft of such a program in GNU Octave. Substitute ??? by appropriate expressions:

```octave
A = tril(1 + round(8 * rand(4, 4)));
disp(A);
s = 0;
for j = 1 : 4
    for k = ??? : ???
        s = s + A(j, k);
    endfor
endfor
disp("My answer:"); disp(s);
disp("Correct answer:"); disp(sum(sum(A)));```

3. Lower triangular summing by rows. Generalize the formula to the case of a $n \times n$ matrix with entries $a_{j,k}$:

$$S = \sum_{j=1}^{n} \sum_{k=1}^{j}.$$

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Lower triangular summing by columns

4. Write the sum corresponding to the picture. Now we are summing by columns.

First write all summands in the explicit form (joining them by columns):

\[ S = \left( a_{1,1} + a_{2,1} + a_{3,1} + a_{4,1} \right) + \left( \right) + \left( \right) + \left( \right). \]

Rewrite each one of the little sums using the symbol \( \sum \):

\[ S = \sum_{j=1}^{4} a_{j,1} + \sum + \sum + \sum. \]

Note that all summands have the same form. Pass to a double sum:

\[ S = \sum_{k=1}^{4} \sum a_{j,k}. \]

5. Write a program that creates a random lower triangular matrix and sums its entries by columns.
Diagonals of a lower triangular matrix

6. Calculate the difference $d := j - k$ for each entry $a_{j,k}$ of the following lower triangular matrix:

7. If $a_{j,k}$ belongs to the second diagonal, then $j$ can be expressed through $k$ in the following manner:

$$j = ?$$

8. Fix some $d \in \{0, 1, 2, 3\}$. If $a_{j,k}$ is on the diagonal with index $d$, then $j$ can be expressed through $k$ and $d$ in the following manner:

$$j = ?$$
Lower triangular summing by diagonals

9. Write the sum corresponding to the picture. Now we are summing by diagonals.

First write all summands in the explicit form (joining them by diagonals):

\[ S = (a_{1,1} + a_{2,2} + a_{3,3} + a_{4,4}) + \left( \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \end{array} \right) + \left( \begin{array}{c} \text{ } \\ \text{ } \end{array} \right) + \left( \begin{array}{c} \text{ } \\ \text{ } \end{array} \right). \]

In each group the first index ("j") can be expressed through the second ("k") by a simple rule, so we can avoid j:

\[ S = \sum_{d=0}^{4} a_{k,k} + \sum_{k=1}^{a_{1,k}} + \sum_{k=1}^{a_{2,k}} + \sum_{k=1}^{a_{3,k}}. \]

The first sum corresponds to \( d = 0 \), the second to \( d = 1 \), and so on:

\[ S = \sum_{d=0}^{4} \sum_{k=1}^{a_{d,k}}. \]
Double lower triangular sum and formal changes of variables

10. Lower triangular summing by rows (recall). Recall the general formula for the lower triangular sum ($n \times n$ case) in which the terms are collected by rows:

$$S = \sum_{j} \sum_{k} a_{j,k}. \quad (1)$$

11. Summing by columns via an interchange of variables. Write two double inequalities that must satisfy $j$ and $k$:

$$\begin{align*}
? \leq j & \leq ?, \\
? \leq k & \leq ?.
\end{align*}$$

Separate these two double inequalities in four simple inequalities:

$$\begin{align*}
? \leq j & \leq ?, \\
? \leq j & \leq ?, \\
? \geq k & \geq ?, \\
? \leq k & \leq ?.
\end{align*}$$

What is the maximum value (independent on $j$) that can take $k$? What is the minimum value (independent on $j$) that can take $k$? In other words, what are global limits of $k$? Find the minimum and maximum values of $k$ (independent on $j$):

$$\begin{align*}
? \leq k & \leq ?.
\end{align*}$$

For each $k$ fixed, find restrictions on $j$:

$$\begin{align*}
? \leq j & \leq ?.
\end{align*}$$

Rewrite the sum $S$:

$$S = \sum_{k=\_}^{\_} \sum_{j=\_}^{\_} a_{j,k}.$$
12. **Summing by diagonals via a formal change of variables.** Write 4 inequalities that satisfy $j$ and $k$ in the sum (1):

$$j \geq \text{?}, \quad j \leq \text{?}, \quad k \geq \text{?}, \quad k \leq \text{?}.$$ 

Find restrictions that satisfy the difference $d$:

$$\text{?} \leq d \leq \text{?}$$

Express $j$ through $d$ and $k$:

$$j = \text{?}.$$ 

Substitute this expression into the equalities with $j$, then solve them for $k$ (treat $d$ as a parameter):

Rewrite the sum $S$ as a double sum over $d$ and $k$:

$$S = \sum_{d=\text{?}} \sum_{k=\text{?}} \sum_{a_{?,k}}.$$