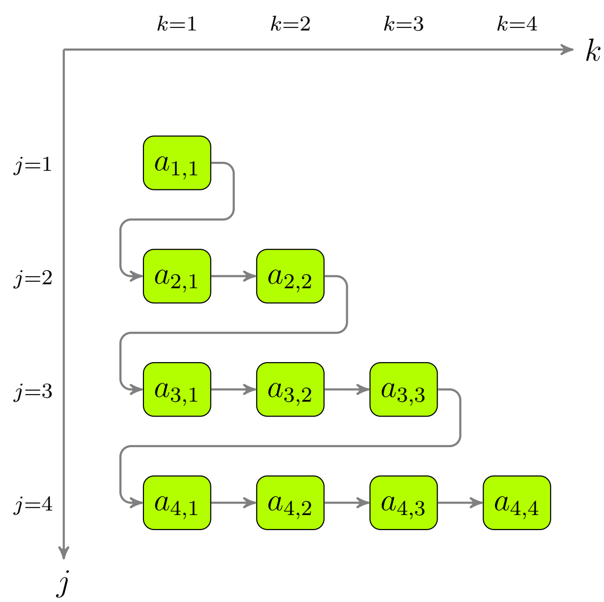
# Lower triangular sums

**Objectives.** Learn to write lower triangular sums in various manners: by rows, by columns and by diagonals.

**Requirements.** Experience of working with notation $\sum$, experience of programming (especially in some language that deals with matrices and starts indices by 1).

## Lower triangular summing by rows




**1.** Consider the sum corresponding to the picture. First we write all summands in the explicit form (joining summands by rows):

$$S = \Big(a_{1,1}\Big) + \Big(a_{2,1} + a_{2,2}\Big) + \Big(\qquad\qquad\qquad\qquad\Big) + \Big(\qquad\qquad\qquad\qquad\qquad\Big).$$

Now we rewrite each one of the little sums using the symbol $\sum$:

$$S = \sum_{k=1}^{1} a_{1,k} + \sum_{k=1}^{2} a_{2,k} + \sum \qquad + \sum \qquad .$$

All summands have the same form, and we can write $S$ as a double sum. Write correctly the limits of the inner sum:

$$S = \sum_{j=1}^{4} \sum_{k=} a_{j,k}.$$

**2. Program (summing by rows the entries of a lower triangular matrix).** Write a simple program that creates a pseudorandom lower triangular matrix and sums all its entries row by row. Here is a draft of such a program in GNU Octave. Substitute ??? by apropiate expressions:

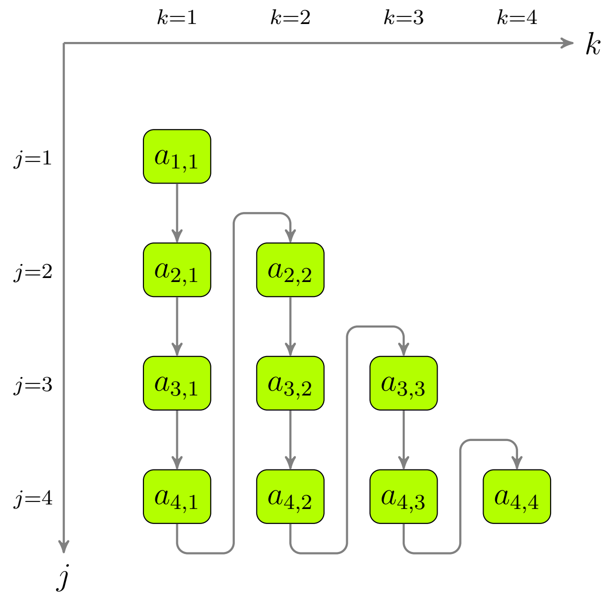
```
A = tril(1 + round(8 * rand(4, 4)));
disp(A);
s = 0;
for j = 1 : 4
    for k = ??? : ???
        s = s + A(j, k);
    endfor
endfor
disp("My answer:"); disp(s);
disp("Correct answer:"); disp(sum(sum(A)));
```

**3. Lower triangular summing by rows.** Generalize the formula to the case of a $n \times n$ matrix with entries $a_{j,k}$:

$$S = \sum_{j=} \sum_{k=} \qquad .$$

## Lower triangular summing by columns

**4.** Write the sum corresponding to the picture. Now we are summing by columns.




First write all summands in the explicit form (joining them by columns):

$$S = \Big(a_{1,1} + a_{2,1} + a_{3,1} + a_{4,1}\Big) + \Big( \qquad\qquad\qquad\qquad \Big) + \Big( \qquad\qquad\qquad \Big) + \Big( \qquad \Big).$$

Rewrite each one of the little sums using the symbol $\sum$:

$$S = \sum_{j=1}^{4} a_{j,1} + \sum \qquad + \sum \qquad + \sum \qquad .$$

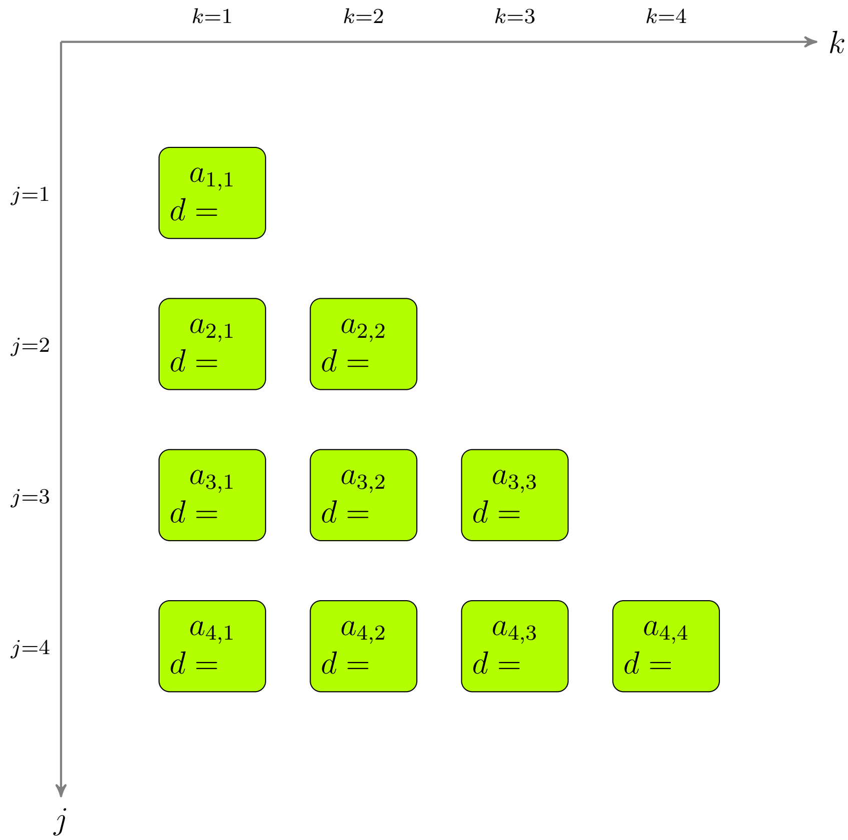
Note that all summands have the same form. Pass to a double sum:

$$S = \sum_{k=}\ \sum_{j=}\ a_{j,k}.$$

**5.** Write a program that creates a random lower triangular matrix and sums its entries by columns.

## Diagonals of a lower triangular matrix

**6.** Calculate the difference $d \coloneqq j - k$ for each entry $a_{j,k}$ of the following lower triangular matrix:




The entries satisfying $j = k$ form the *main diagonal* (or the *zeroth diagonal*) of the matrix; the entries satisfying $j - k = 1$ are on the *first diagonal*, and so on.

**7.** If $a_{j,k}$ belongs to the second diagonal, then $j$ can be expressed through $k$ in the following manner:

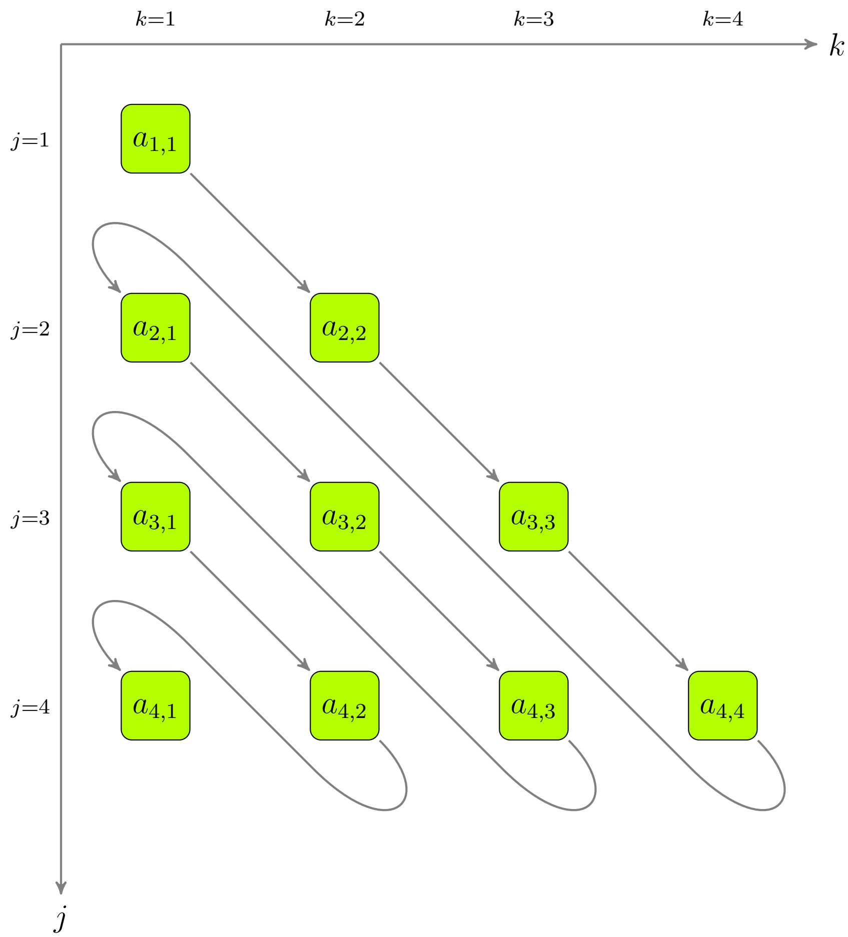
$$j = \underbrace{\qquad\qquad}_{?}.$$

**8.** Fix some $d \in \{0, 1, 2, 3\}$. If $a_{j,k}$ is on the diagonal with index $d$, then $j$ can be expressed through $k$ and $d$ in the following manner:

$$j = \underbrace{\qquad\qquad}_{?}.$$

## Lower triangular summing by diagonals

**9.** Write the sum corresponding to the picture. Now we are summing by diagonals.




First write all summands in the explicit form (joining them by diagonals):

$$S = \left(a_{1,1} + a_{2,2} + a_{3,3} + a_{4,4}\right) + \left( \qquad\qquad\qquad \right) + \left( \qquad\qquad \right) + \left( \qquad \right).$$

In each group the first index ("$j$") can be expressed through the second ("$k$") by a simple rule, so we can avoid $j$:

$$S = \sum_{k=1}^{4} a_{k,k} + \sum_{k=} \underbrace{\qquad\qquad}_{a_{?,k}} + \sum_{k=} \qquad\qquad + \sum_{k=} \qquad\qquad .$$

The first sum corresponds to $d = 0$, the second to $d = 1$, and so on:

$$S = \sum_{d=} \sum_{k=} \qquad\qquad .$$

## Double lower triangular sum and formal changes of variables

**10. Lower triangular summing by rows (recall).** Recall the general formula for the lower triangular sum ($n \times n$ case) in which the terms are collected by rows:

$$S = \sum_{j=} \sum_{k=} a_{j,k}. \tag{1}$$

**11. Summing by columns via an interchange of variables.** Write two double inequalities that must satisfy $j$ and $k$:

$$\underbrace{}_{?} \le j \le \underbrace{}_{?}, \qquad \underbrace{}_{?} \le k \le \underbrace{}_{?}.$$

Separate these two double inequalities in four simple inequalities:

$$j \ge \underbrace{}_{?}, \qquad j \le \underbrace{}_{?}, \qquad k \ge \underbrace{}_{?}, \qquad k \le \underbrace{}_{?}.$$

What is the maximum value (independent on $j$) that can take $k$?
What is the minimum value (independent on $j$) that can take $k$?
In other words, what are global limits of $k$?
Find the minimum and maximum values of $k$ (independent on $j$):

$$\underbrace{}_{?} \le k \le \underbrace{}_{?}.$$

For each $k$ fixed, find restrictions on $j$:

$$\underbrace{}_{?} \le j \le \underbrace{}_{?}.$$

Rewrite the sum $S$:

$$S = \sum_{k=} \sum_{j=} a_{j,k}.$$

**12. Summing by diagonals via a formal change of variables.** Write 4 inequalities that satisfy $j$ and $k$ in the sum (1):

$$j \geq \underbrace{\qquad}_{?}, \qquad j \leq \underbrace{\qquad}_{?}, \qquad k \geq \underbrace{\qquad}_{?}, \qquad k \leq \underbrace{\qquad}_{?}.$$

Find restrictions that satisfy the difference $d$:

$$\underbrace{\qquad\quad}_{?} \leq d \leq \underbrace{\qquad\quad}_{?}$$

Express $j$ through $d$ and $k$:

$$j = \underbrace{\qquad\quad}_{?}.$$

Substitute this expression into the equalities with $j$, then solve them for $k$ (treat $d$ as a parameter):

Rewrite the sum $S$ as a double sum over $d$ and $k$:

$$S = \sum_{d=} \sum_{k=} \underbrace{\qquad\qquad}_{a_{?,k}}.$$